Newberger Math 247 Spring 03

Quiz 4.4 and 4.7 name:__________________________

1. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V and suppose $\mathbf{a}_1 = 2\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3$, $\mathbf{a}_2 = 3\mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = -3\mathbf{b}_1 + \mathbf{b}_2$.

a. (4 points) Find the change of coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

The columns of the change of coordinate matrix $\underset{\mathcal{B}\leftarrow\mathcal{A}}{P}$ *from* $\mathcal{A}$ *to* $\mathcal{B}$ *are the vectors* $[\mathbf{a}_1]_\mathcal{B}$, $[\mathbf{a}_2]_\mathcal{B}$, *and* $[\mathbf{a}_3]_\mathcal{B}$. *The coordinates* $[\mathbf{a}_1]_\mathcal{B}$ *of* $\mathbf{a}_1$ *relative to* $\mathcal{B}$ *are the weights when* $\mathbf{a}_1$ *is written as a linear combination of the vectors in* $\mathcal{B}$. *Thus since* $\mathbf{a}_1 = 2\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3$, *we have*

$$[\mathbf{a}_1]_\mathcal{B} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Similarly,

$$[\mathbf{a}_2]_\mathcal{B} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix},$$

and

$$[\mathbf{a}_3]_\mathcal{B} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

Thus

$$\underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

b. (4 points) Refereing to the bases $\mathcal{A}$ and $\mathcal{B}$ above, find $[\mathbf{x}]_\mathcal{B}$ where $\mathbf{x} = \mathbf{a}_1 - 2\mathbf{a}_2 + 2\mathbf{a}_3$.

The coordinates $[\mathbf{x}]_\mathcal{A}$ *of* $\mathbf{x}$ *relative to* $\mathcal{A}$ *are the weights when* $\mathbf{x}$ *is written as a linear combination of the vectors in* $\mathcal{A}$. *Thus*

$$[\mathbf{x}]_\mathcal{A} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Now to calculate $[\mathbf{x}]_\mathcal{B}$ *we use the change of basis matrix we found in part (a):*

$$\begin{aligned} [\mathbf{x}]_\mathcal{B} &= \underset{\mathcal{B}\leftarrow\mathcal{A}}{P}[\mathbf{x}]_\mathcal{A} \\ &= \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -5 \\ -1 \end{bmatrix}. \end{aligned}$$

2. Let $\mathcal{F} = \left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}$, and let $\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

a. (4 points) Find $\underset{\mathcal{F}\leftarrow\mathcal{D}}{P}$.

To find $\underset{\mathcal{F}\leftarrow\mathcal{D}}{P}$ *we row reduce:*

$$\left[\begin{array}{cc|cc} 5 & 0 & 1 & 1 \\ 0 & 6 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & \frac{1}{6} \end{array}\right]$$

Thus

$$\underset{\mathcal{F}\leftarrow\mathcal{D}}{P} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{6} \end{bmatrix}.$$

b. (4 points) Find $\underset{\mathcal{D}\leftarrow\mathcal{F}}{P}$.

To find $\underset{\mathcal{D}\leftarrow\mathcal{F}}{P}$, *we calculate the inverse of* $\underset{\mathcal{F}\leftarrow\mathcal{D}}{P}$. *We use the formula:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Thus we get

$$\underset{\mathcal{D}\leftarrow\mathcal{F}}{P} = \frac{1}{1/30} \begin{bmatrix} \frac{1}{6} & -\frac{1}{5} \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 0 & 6 \end{bmatrix}.$$

3. a. (2 points) Vector space operations in $\mathbb{P}^3$ correspond to vector space operations in $\mathbb{R}^k$ where $\underline{k = 4}$.

b. (2 points) What is the coordinate vector of $1 - 2t^2 + t^4$ with respect to the standard basis $\mathcal{S} = \{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}_4$?

The entries in the coordinate vector of $1 - 2t^2 + t^4$ *relative to the basis* $\mathcal{S} = \{1, t, t^2, t^3, t^4\}$ *are the coefficients:*

$$[1 - 2t^2 + t^4]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

c. (4 points) Let $\mathcal{B} = \{1 + 5t^2, t + t^2, 1 - 3t + 2t^2\}$. Is $\mathcal{B}$ a linearly independent set of polynomials?

First we write the coordinate vectors for the polynomials in $\mathcal{B}$. *We get*

$$\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

Since $\mathbb{R}^3$ *is isomorphic to* $\mathbb{P}^2$, *it is enough to check if these vectors are independent in* $\mathbb{R}^3$. *We row reduce:*

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is a column without a pivot, these vectors are not linearly independent, so the polynomials in $\mathcal{B}$ *are not linearly independent either.*