Newberger Math 247 Spring 03

Quiz 1.8 name:________________________

1. Let T be the transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) (4 points) What must a and b be so that $T : \mathbb{R}^a \to \mathbb{R}^b$?

The number of entries in $\mathbf{x}$ *must match the number of columns of* A *for the definition* $T(\mathbf{x}) = A\mathbf{x}$ *to make sense. So* $a = 2$*. Similarly, since* $A\mathbf{x}$ *is a linear combination of the columns of* A*,* $T(\mathbf{x})$ *is a linear combination of the columns of* A*, and hence for each* $\mathbf{x}$*,* $T(\mathbf{x})$ *is a vector with the same number of rows as* A*. So* $b = 4$*.*

(b) (5 points) Let $\mathbf{b} = \begin{bmatrix} 3 \\ 12 \\ 3 \\ 0 \end{bmatrix}$. Is $\mathbf{b}$ in the range of T? How do you know?

We want to know if there is a vector $\mathbf{x}$ *such that* $T(\mathbf{x}) = \mathbf{b}$*, so we solve the equation* $A\mathbf{x} = \mathbf{b}$*, given by the augmented matrix below.*

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since this system is consistent, there is at least one vector $\mathbf{x}$ *such that* $A\mathbf{x} = \mathbf{b}$*, and hence by the definition of* T*, there is at least one vector* $\mathbf{x}$ *such that* $T(\mathbf{x}) = \mathbf{b}$*, which means that* $\mathbf{b}$ *is in the range of* T*.*

(c) (5 points) Does every vector in the codomain of T lie in the range of T for the transformation T given above? How do you know?

No. Since A *does not have a pivot position in every row, there are vectors* $\mathbf{c}$ *so that* $A\mathbf{x} = \mathbf{c}$ *does not have a solution. Such a vector* $\mathbf{c}$ *lies in the codomain of* T*, but not in the range of* T*.*

2. Determine whether or not each of the given transformations are linear. Give reasons that show your answer is correct.

(a) (5 points) $T(x_1, x_2) = (-4x_2, x_1 + x_2)$.

We show T *is linear by writing* T *as a matrix transformation. Note that*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -4x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} -4x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -4 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus T *is a matrix transformation, and we know* T *is linear since all matrix transformations are linear.*

(b) (5 points) $T(x_1, x_2) = (-4, x_1 + x_2)$.

If we were to try to make a matrix transformation, we would have a problem with the -4 *in the first entry. Thus we should be inclined to believe that the matrix is not linear. We can justify that* T *is not linear in various ways. The easiest is to note that* $T(\mathbf{0}) = (-4, 0) \neq \mathbf{0}$*, and since a linear transformation must satisfy* $T(\mathbf{0}) = \mathbf{0}$*, we know* T *is not linear.*

Another way to explain that the transformation is not linear is to show that one of the statements in the definition of linear is false for some particular vectors. For example, we could show that there are vectors $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$*. In fact any choice of* $\mathbf{u}$ *and* $\mathbf{v}$ *would work: take* $\mathbf{u} = (1, 1)$ *and* $\mathbf{v} = (0, 0)$*. Then we get that the right hand side is* $T(1, 1) + T(0, 0) = (-4, 2) + (-4, 0) = (-8, 2)$ *while the left hand side is* $T((1, 1) + (0, 0)) = T(1, 1) = (-4, 2)$*. Thus the equation* $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *does not hold for our choice of* $\mathbf{u}$ *and* $\mathbf{v}$*, and* T *does not satisfy the definition of linear.*