Newberger Math 247 Spring 02

Solutions for Quiz 1.3 and 1.4

Let

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \mathbf{u}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}.$$

1. (12 points) Show that $\mathbf{b}$ is in $Span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. In other words show that $\mathbf{b}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

   We want to show we can find numbers x_1, x_2 and x_3, such that

   $$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{b}.$$

   To see this we reduce the following augmented matrix to echelon form:

   $$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & 1 \\ 0 & 2 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   This system is consistent, so $\mathbf{b}$ is in $Span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

2. (4 points) Name three real numbers, x_1, x_2 and x_3, such that

   $$\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3.$$

   In (1), we showed the numbers x_1, x_2 and x_3 exist; in this problem we are asked to find such numbers. To solve for x_1, x_2 and x_3, put the matrix from (1) in reduced echelon form (as it turns out, in this example it is already in reduced echelon form!

   $$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   From this we see that the solution of this system is

   $$x_1 = 2 - 5x_3$$
   $$x_2 = 3 - 4x_3$$
   $$x_3 \text{ is free.}$$

   Thus for any choice of x_3 we will get a solution of our system. For example, $x_1 = 2$, $x_2 = 3$ and $x_3 = 0$ is a solution, and we get

   $$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}.$$

   Any example of a solution (given by any choice of x_3) would be a correct answer to this question.

3. (8 points) Let A be the matrix whose columns are $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. In other words, let $A = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. Do the columns of A span all of $\mathbb{R}^3$? Why or why not?

To answer this question we reduce the matrix A to echelon form; fortunately we have already done this in answering question (1).

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since A does not have a pivot in every row, the columns of A do not span all of $\mathbb{R}^3$; in other words, not all vectors in $\mathbb{R}^3$ are linear combinations of the columns of A. The above is a correct complete answer to that quiz question; I include the following as an extra resource on this topic. What if we were asked, "What condition implies that a vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does lie in the span of the columns of A?" To answer this, we must reduce the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & b_1 \\ -2 & 1 & -6 & b_2 \\ 0 & 2 & 8 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & b_1 \\ 0 & 1 & 4 & 2b_1 + b_2 \\ 0 & 2 & 8 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & -2(2b_1 + b_2) + b_3 \end{array}\right].$$

We see that the only way this can be consistent is for $-2(2b_1 + b_2) + b_3$ to equal 0. So a vector $\mathbf{b}$ is in the span of the columns of A if and only if we have $-2(2b_1 + b_2) + b_3 = 0$.