Newberger Math 247 Spring 03

Homework solutions: Section 4.1 #5-8

In Exercises 5-8 determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answer.

5. All polynomials of the form $\mathbf{p}(t) = at^2$.

The set of all polynomials of the form $\mathbf{p}(t) = at^2$ *is all scalar multiples of the polynomial* t^2*, thus this set is just Span*$\{t^2\}$*. Since all spans are subspaces, this set is a subspace.*

6. All polynomials of the form $\mathbf{p}(t) = a + t^2$.

Let H *denote the set of all polynomials of the form* $\mathbf{p}(t) = a + t^2$*. Then we can check that the zero polynomial* $0 + 0t + 0t^2$ *is not in* H*. We try to find a number* a *such that* $a + t^2 = 0 + 0t + 0t^2$*. Since the coefficient on* t^2 *on the LHS is 1, but the coefficient on* t^2 *on the RHS is 0, these can never be equal (since* $0 \neq 1$*). Thus the zero polynomial is not in* H*, since there is no value of* a *that will make* $a+t^2 = 0+0t+0t^2$*. Since* H *does not contain the zero polynomial,* H *is not a subspace.*

7. All polynomials of degree at most 3, with integers as coefficients.

Let H *denote the set of all polynomials of degree at most 3, with integers as coefficients. Then the zero polynomial is in* H *since 0 is an integer. Furthermore, if we take* $\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ *and* $\mathbf{q}(t) = b_0+b_1t+b_2t^2+b_3t^3$ *in* H*, then* $a_0, ..., a_3$ *and* $b_0, ..., b_3$ *are integers. So*

$$\mathbf{p}(t) + \mathbf{q}(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3$$

is in H *since* (a_0+b_0)*,* (a_1+b_1)*,* (a_2+b_2)*, and* (a_3+b_3) *are all integers. So* H *is closed under addition.*

However, H *is not a subspace of* $\mathbb{P}_3$ *since* H *is not closed under scalar multiplication. For example, let* $\mathbf{p}(t) = 1 + t$*. Then* $\mathbf{p}$ *is in* H *since the coefficients are* 1*,* 1*,* 0 *and* 0*, which are all integers. However if* $c = \sqrt{2}$*, then* $c\mathbf{p}(t) = \sqrt{2}+\sqrt{2}t$ *is not in* H *since* $\sqrt{2}$ *is not an integer.*

8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

Let H *denote the set of all polynomials in* $\mathbb{P}_n$ *such that* $\mathbf{p}(0) = 0$*. Suppose* $\mathbf{p}(t) = a_0+a_1t+\cdots+a_nt^n$ *is an element of* H*. Then* $\mathbf{p}(0) = 0$*. But* $\mathbf{p}(0) = a_0$*, so we get that* H *is all polynomials of the form* $\mathbf{p}(t) =$

$a_0 + \cdots + a_n t^n$ where $a_0 = 0$. In other words, H is all polynomials of the form $\mathbf{p}(t) = a_1 t + \cdots + a_n t^n$. But this shows that H is just all linear combinations of the vectors $\{t, t^2, \ldots, t^n\}$, i.e. $H =$Span$\{t, t^2, \ldots, t^n\}$. So H is a subspace, since all spans are subspaces.

One additional example:

All polynomials of the form $\mathbf{p}(t) = a + bt + 3at^2$.

Let H be the set of all polynomials of the form $\mathbf{p}(t) = a + bt + 3at^2$. Then H is all polynomials of the form $\mathbf{p}(t) = a(1 + 3t^2) + bt$. In other words, H is all linear combinations of $1 + 3t^2$ and t. So

$$H = Span\{1 + 3t^2, t\},$$

and H is a subspace of $\mathbb{P}_2$.