Newberger Math 247 Spring 03

Homework solutions: Section 2.1 #18,20-24, Section 2.2 #14, 16, 18, 20 and Section 2.3 #16, 18, 22

Section 2.1 #18,20-24.

18. Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of B are equal. What can you say about the columns of AB (if AB is defined)? Why?

We start with the assumption (the "Suppose" part) that $\mathbf{b}_1 = \mathbf{b}_2$. *So* $A\mathbf{b}_1 = A\mathbf{b}_2$. *Now we know* $AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_n]$, *so we see that the first two columns of* AB *are also equal.*

20. Suppose the second column of B is all zeros. What can you say about the second column of AB?

We start with the assumption that $\mathbf{b}_2 = \mathbf{0}$. *So* $A\mathbf{b}_2 = A\mathbf{0} = \mathbf{0}$. *Now we know* $AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_n] = [A\mathbf{b}_1, \mathbf{0}, \ldots, A\mathbf{b}_n]$, *so we see that the second column of* AB *is all zeros as well.*

21. Suppose the last column of AB is entirely zero but B itself has no column of zeros. What can you say about the columns of A?

Let $B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$. *Then* $AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_n]$. *We start with the assumption that the last column of* AB *is the zero vector. So we get that* $A\mathbf{b}_n = \mathbf{0}$. *Thus* $\mathbf{x} = \mathbf{b}_n$ *is a solution to the equation* $A\mathbf{x} = \mathbf{0}$. *Since* B *has no columns that are entirely zero, we know* $\mathbf{b}_n \neq \mathbf{0}$, *so* $\mathbf{x} = \mathbf{b}_n$ *is a non-trivial solution to the equation* $A\mathbf{x} = \mathbf{0}$. *Since the homogeneous equation* $A\mathbf{x} = \mathbf{0}$ *has non-trivial solutions, the columns of* A *are linearly dependent.*

22. Show that if the columns of B are linearly dependent, then so are the columns of AB.

You might want to reword this problem into the "Suppose..., Explain why..." form that you are used to. In that form the statement would be: Suppose that the columns of B *are linearly dependent. Explain why the columns of* AB *are also independent.*

Now we begin with the assumption (the "Suppose" part) that the columns of B *are linearly dependent. One way for the columns of* B *to be linearly dependent is for one column to be a multiple of another column. However we cannot assume that that is the case, because there are other relationships between the columns of* B *that would cause the columns to be linearly dependent. Theorem 7 on page 68 gives us a characterization of linearly dependent sets that is always true.*

Since the columns of B are linearly dependent, one of the columns, say $\mathbf{b}_p$ can be written as a linear combination of the columns $\mathbf{b}_1, \ldots \mathbf{b}_{p-1}$. Thus there are scalars, $c_1, \ldots, c_{p-1}$, such that $\mathbf{b}_p = c_1\mathbf{b}_1 + \cdots + c_{p-1}\mathbf{b}_{p-1}$. Now we know $AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_n]$, so the p^{th} column of AB is

$$A\mathbf{b}_p = A(c_1\mathbf{b}_1 + \cdots + c_{p-1}\mathbf{b}_{p-1}) = c_1A\mathbf{b}_1 + \cdots + c_{p-1}A\mathbf{b}_{p-1}.$$

Thus the p^{th} column of AB is a linear combination of the previous $p-1$ columns of AB, and hence the columns of AB are linearly independent.

23. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

We want to know what the solutions to $A\mathbf{x} = \mathbf{0}$ are. We are given that $CA = I$. To combine these we need to either get an $\mathbf{x}$ in the second equation or a C in the first equation. We will get a C in the first equation by multiplying on the left by C. We get $A\mathbf{x} = \mathbf{0}$ implies $CA\mathbf{x} = C\mathbf{0}$. But the LHS of this is $CA\mathbf{x} = I\mathbf{x} = \mathbf{x}$ and the RHS of this is $C\mathbf{0} = \mathbf{0}$. So we get that whenever we have $A\mathbf{x} = \mathbf{0}$, we must have $\mathbf{x} = \mathbf{0}$. Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. We know that this means that there is a pivot in every column of A, and A cannot have more columns than rows. So we see that if there exists a matrix C with $CA = I$, then A has at least as many rows as columns.

24. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why A cannot have more rows than columns.

For a vector $\mathbf{b}$ in $\mathbb{R}^m$, we want to find $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. We know $AD = I_m$. If we put $\mathbf{x} = D\mathbf{f}$, then we get $A\mathbf{x} = A(D\mathbf{b}) = (AD)\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Thus $\mathbf{x} = D\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Now we see that for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution, so the columns of A span $\mathbb{R}^m$, and A has a pivot position in every row. This shows that there cannot be more rows than columns in A. Thus whenever there is a matrix D such that $AD = I_m$, A has at least as many columns as rows.

Section 2.2 #14, 16, 18, 20

14. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

We know D is invertible so we have the matrix D^{-1} to work with. We have

$$(B - C)D = 0$$

so

$$BD - CD = 0$$

by the distributive law (do not switch the order of the matrices!). So we have

$$BD = CD$$

by adding CD to both sides of the equation. Now multiply the equation by D^{-1} **on the right**. *We get*

$$(BD)D^{-1} = (CD)D^{-1},$$

and applying the associative law, this means

$$B(DD^{-1}) = C(DD^{-1}).$$

Now $DD^{-1} = I$ so we get

$$BI = CI,$$

which gives us $B = C$.

16. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

We will show that A is the product of invertible matrixes, so it is invertible by Theorem 6b. Let $C = AB$. Then $CB^{-1} = (AB)B^{-1} = A(BB^{-1}) = AI = A$. Since B is invertible, so is B^{-1}, and C is invertible by assumption. Thus A is invertible.

18. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

Since P is invertible, we can use P^{-1} in our calculation. We have

$$A = PBP^{-1},$$

so

$$P^{-1}A = P^{-1}PBP^{-1},$$

which implies

$$P^{-1}A = BP^{-1},$$

and hence

$$P^{-1}AP = BP^{-1}P,$$

which implies

$$P^{-1}A = B.$$

20. Suppose A, B and X are $n \times n$ matrices such that A, X and $(A - AX)$ are invertible and suppose

$$(A - AX)^{-1} = X^{-1}B \tag{1}$$

a. Explain why B is invertible.
b. Solve Equation (1) for X.

a. Multiply the equation

$$(A - AX)^{-1} = X^{-1}B$$

by X on the left, to get

$$X(A - AX)^{-1} = XX^{-1}B,$$

which implies

$$X(A - AX)^{-1} = B.$$

Thus B is a product of X and $(A - AX)^{-1}$, both of which are invertible by assumption. So B is invertible since it is the product of invertible matrices. Note that since $(A - AX)$ is invertible, $(A - AX)^{-1}$ is also invertible.

b. We will solve the equation

$$(A - AX)^{-1} = X^{-1}B$$

for X using matrix algebra. First we multiply on the right by $(A - AX)$. We get

$$I = X^{-1}B(A - AX).$$

Now multiply on the left by X. We get

$$X = B(A - AX).$$

By the distributive law we get

$$X = BA - BAX,$$

and adding BAX to both sides, this becomes

$$BAX + X = BA.$$

Thus by the distributive law

$$(BA + I)X = BA.$$

Now we want to multiply by $(BA + I)^{-1}$ to leave the X by itself; but to do this we must say how we know $(BA + I)^{-1}$ exists. Note that the equation we have implies (by multiplying on the right by X^{-1})

$$(BA + I) = BAX^{-1},$$

so $BA + I$ is a product of B, A and X^{-1}. Each of these are invertible, so we have that $BA + I$ is invertible as well. Now we multiply our equation on the left by $(BA + I)^{-1}$, getting

$$X = (BA + I)^{-1}BA.$$

Section 2.3 #16, 18, 22.

16. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

It is not possible since if the matrix is invertible, there is a pivot in every row and the columns must span $\mathbb{R}^5$.

You may also answer this question by referring to Theorem 8. Since the matrix is invertible, item (a) of Theorem 8 holds, so all the other items hold. In particular, item (h) holds, and the columns span $\mathbb{R}^6$.

18. If C is 6×6 and the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^6$, is it possible that for some $\mathbf{v}$, $C\mathbf{x} = \mathbf{v}$ has more than one solution? Why or why not?

It is not possible that for some $\mathbf{v}$, $C\mathbf{x} = \mathbf{v}$ has more than one solution. Since the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^6$, C has a pivot in every row. This means it also has a pivot in every column, since the matrix is square. Thus there can be no free variables, and for all $\mathbf{v}$ the equation $C\mathbf{x} = \mathbf{v}$ will have a unique solution.

You may also answer this question by referring to Theorem 8. Since $C\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^7$, item (g) in Theorem 8 holds, so all of the other items also hold. In particular, item (d), which says the equation $C\mathbf{x} = \mathbf{0}$ has only the trivial solution, and there are no free variables.

22. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$, what can you say about the equation $H\mathbf{x} = \mathbf{0}$?

If the equation $H\mathbf{x} = \mathbf{c}$ *is inconsistent for some* $\mathbf{c}$ *in* $\mathbb{R}^n$*, then there must be a row without a pivot. Thus there is a column without a pivot and a free variable. This implies* $H\mathbf{x} = \mathbf{0}$ *has non-trivial solutions.*

You may also answer this question by referring to Theorem 8. Since the equation $H\mathbf{x} = \mathbf{c}$ *is inconsistent for some* $\mathbf{c}$ *in* $\mathbb{R}^n$*, item (g) of Theorem 8 fails, and so all of the rest of the items fail. In particular, item (d) fails, and* $H\mathbf{x} = \mathbf{0}$ *must have non-trivial solutions.*