Newberger Math 247 Spring 03

Homework solutions: Section 1.8 #26, 27, 31, Section 2.3 #34,36,37,38

Section 1.8 #26,27,31.

26. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbb{R}^3$ and let P be the plane through $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s and t in $\mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps P onto a plane through $\mathbf{0}$, a line through $\mathbf{0}$ or onto just the origin $\mathbf{0}$ in $\mathbb{R}^3$. What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?

We want to know the image of P under T. Any point $\mathbf{x}$ on P will have the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ where s and t are real numbers. So $T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v})$, and $T(P)$ is the collection of all vectors of the form $sT(\mathbf{u})+tT(\mathbf{v})$ where s and t are any real number. Thus the image of P under T is Span$\{T(\mathbf{u}), T(\mathbf{v})\}$.

Since there are three cases for the geometric description of the span of two vectors in $\mathbb{R}^3$, there are three cases for the geometric description of the image of P under T. First $\{T(\mathbf{u}), T(\mathbf{v})\}$ could be independent, in which case, the image of P under T is a plane. Second, $\{T(\mathbf{u}), T(\mathbf{v})\}$ could be dependent but not both zero, in which case the image of P is a line. Third, both vectors $T(\mathbf{u})$ and $T(\mathbf{v})$ could be zero, in which case the image of P is simply $\mathbf{0}$.

27. a. Show that the line through vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ may be written in parametric form $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$.

b. The line segment from $\mathbf{p}$ to $\mathbf{p}$ is the set of points of the form $(1-t)\mathbf{p}+t\mathbf{q}$ where $0 \leq t \geq 1$. Show that a linear transformation T maps this line segment onto a line segment or onto a single point.

a. The line through vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ is parallel to the vector $\mathbf{q}-\mathbf{p}$ and through $\mathbf{p}$. Thus it has parametric vector form $(\mathbf{q}-\mathbf{p})t + \mathbf{p} = (1-t)\mathbf{p} + t\mathbf{q}$.

b. Apply the transformation to the line segment. We get

$$T((1-t)\mathbf{p} + t\mathbf{q}) = (1-t)T(\mathbf{p}) + tT(\mathbf{q})$$

since T is linear. Thus there are two cases. First if $T(\mathbf{p}) = T(\mathbf{q})$ then this expression simplifies to $T(\mathbf{p})$, which is a single point. Second if $T(\mathbf{p}) \neq T(\mathbf{q})$, then the expression describes a line segment parallel to the vector $T(\mathbf{q}) - T(\mathbf{p})$ and through $T(\mathbf{p})$.

31. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, the homogeneous equation

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{x} = \mathbf{0}$$

has non-trivial solutions. Let $\mathbf{c}$ be one of these non-trivial solutions. Then

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{c} = \mathbf{0},$$

so

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Now we are interested in $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$*. From the above we get that*

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}).$$

But this means

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0},$$

so the homogeneous equation

$$[T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)]\mathbf{x} = \mathbf{0}$$

has $\mathbf{c}$ *as a solution as well, i.e. it has non trivial solutions. Thus the vectors* $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ *are linearly dependent.*

Section 2.3 #36, 37, 38.

36. Let T be a linear transformation that maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Show that T^{-1} exists and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Is T^{-1} also one-to-one?

Since T *is linear there is a matrix* A *such that* $T(\mathbf{x}) = A\mathbf{x}$*. Since* T *is onto,* A *is invertible by Theorem 8. By Theorem 9,* T^{-1} *exists and equals* $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$*. Since* A^{-1} *is the standard matrix for* T^{-1}*, and* A^{-1} *is invertible,* T^{-1} *is one-to-one by the invertible matrix theorem.*

37. Suppose T and U are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(U(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Is it true that $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$?

It is true. Since T *and* U *are linear, there are matrices* A *and* B *such that* $T(\mathbf{x}) = A\mathbf{x}$ *and* $U(\mathbf{x}) = B\mathbf{x}$*. Thus we have that* $T(U(\mathbf{x})) = AB\mathbf{x} = \mathbf{x}$ *for all* $\mathbf{x}$ *in* $\mathbb{R}^n$*. This means* $AB = I$*. Now the theorem at the top of p130 (blue box) states that* A *and* B *are inverses. This means* $BA = I$ *as well. So* $U(T(\mathbf{x})) = BA\mathbf{x} = I\mathbf{x} = \mathbf{x}$*, so* $U(T(\mathbf{x})) = \mathbf{x}$*.*

38. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$?

T *cannot be onto. Since* T *is linear there is a matrix* A *such that* $T(\mathbf{x}) = A\mathbf{x}$*. Since* $T(\mathbf{u}) = T(\mathbf{v})$*, we have* $A\mathbf{u} = A\mathbf{v}$*, which implies* $A\mathbf{u} - A\mathbf{v} = \mathbf{0}$*, or equivalently* $A(\mathbf{u}-\mathbf{v}) = \mathbf{0}$*. Since* $\mathbf{u}$ *and* $\mathbf{v}$ *are distinct,* $\mathbf{u} \neq \mathbf{v}$*, and so* $\mathbf{u}-\mathbf{v} \neq \mathbf{0}$*. Thus* $A\mathbf{x} = \mathbf{0}$ *has a non-trivial solution, namely* $\mathbf{x} = \mathbf{u} - \mathbf{v}$*. This means that* A *is not invertible, and hence* T *is not onto, by the invertible matrix theorem.*