Newberger Math 247 Spring 02

Homework solutions: Section 1.5 #26-32 and Section 1.7 #23-29

Section 1.5 #26-32.

26. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$A\mathbf{x} = \mathbf{0}$ has only the trivial solution when it has a unique solution, which means there are no free variables in the system and so there is a pivot position in every column of the matrix. Similarly, $A\mathbf{x} = \mathbf{b}$ has a unique solution precisely when the system has no free variables, which means the matrix has a pivot position in every column.

27. Suppose A is the 3×3 zero matrix. Describe the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

We solve the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are no pivots so all three variables are free. We get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3,$$

where x_1, x_2 and x_3 are any real numbers. Thus any vector $\mathbf{x}$ in $\mathbb{R}^3$ is a solution to this equation.

28. If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin?

Saying the solution set is a plane through the origin means that the origin, $\mathbf{x} \neq \mathbf{0}$, is a solution. But this means $\mathbf{x} \neq \mathbf{0}$ satisfies $A\mathbf{x} = \mathbf{b}$, in other words, this means that $A\mathbf{0} = \mathbf{b}$. But $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$, so $\mathbf{0}$ is not in the solution set, and the solution set cannot be a plane through the origin.

#29-32: (a) does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution, and (b) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

29. A is a 3×3 matrix with three pivot positions.

This matrix will have echelon form:

$$\begin{pmatrix} \square & \star & \star \\ 0 & \square & \star \\ 0 & 0 & \square \end{pmatrix}$$

(a) *No, $A\mathbf{x} = \mathbf{0}$ does not have a nontrivial solution, because the matrix A has a pivot position in every column, so there will be no free variables.*

(b) *Yes, $A\mathbf{x} = \mathbf{b}$ will have a solution for all $\mathbf{b}$, because A has a pivot position in every row, so no vector $\mathbf{b}$ could cause a row of the form $(0\ 0\ 0\ h)$ where $h \neq 0$ in the augmented matrix.*

30. A is a 3×3 matrix with two pivot positions.
This matrix will have echelon form:

$$\begin{bmatrix} \square & \star & \star \\ 0 & \square & \star \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \square & \star & \star \\ 0 & 0 & \square \\ 0 & 0 & 0 \end{bmatrix}$$

(a) *Yes, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solutions, because the matrix A does not have a pivot position in every column, so there will be a free variable.*
(b) *No, $A\mathbf{x} = \mathbf{b}$ will not have a solution for all $\mathbf{b}$, because A does not have a pivot position in every row, so there are vectors $\mathbf{b}$ that could cause a row of the form $(0\ 0\ 0\ h)$ where $h \neq 0$.*

31. A is a 3×2 matrix with two pivot positions.
This matrix will have echelon form:

$$\begin{pmatrix} \square & \star \\ 0 & \square \\ 0 & 0 \end{pmatrix}$$

(a) *No, $A\mathbf{x} = \mathbf{0}$ does not have a nontrivial solution, because the matrix A has a pivot position in every column, so there will be no free variables.*
(b) *No, $A\mathbf{x} = \mathbf{b}$ will not have a solution for all $\mathbf{b}$, because A does not have a pivot position in every row, so there are vectors $\mathbf{b}$ that could cause a row of the form $(0\ 0\ h)$ where $h \neq 0$.*

32. A is a 3×2 matrix with two pivot positions.
This matrix will have echelon form:

$$\begin{bmatrix} \square & \star & \star \\ 0 & \square & \star \end{bmatrix} \text{ or } \begin{bmatrix} \square & \star & \star \\ 0 & 0 & \square \end{bmatrix}$$

(a) *Yes, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solutions, because the matrix A does not have a pivot position in every column, so there will be a free variable.*
(b) *Yes, $A\mathbf{x} = \mathbf{b}$ will have a solution for all $\mathbf{b}$, because A has a pivot position in every row, so no vector $\mathbf{b}$ could cause a row of the form $(0\ 0\ 0\ h)$ where $h \neq 0$.*

Section 1.7 #23-29.
Instructions for #23-26: Describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2
23. A is a 3×3 matrix with linearly independent columns.
Since the columns are linearly independent, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, so the system cannot have any free variables. This means that there is a pivot position in every column of the matrix A, so the echelon form of A will be:

$$\begin{bmatrix} \square & \star & \star \\ 0 & \square & \star \\ 0 & 0 & \square \end{bmatrix}$$

24. A is a 2×2 matrix with linearly dependent columns.
Since the columns are linearly dependent, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, so the system has free variables. This means that the

matrix A will not have a pivot position in every column. Since there are two columns in A, this means there will either be one or no pivot positions, and the echelon form will be one of the following:

$$\begin{bmatrix} \square & \star \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \square \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

25. A is a 4×2 matrix $A = [\mathbf{a}_1, \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

Since $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$, the set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and so A has a pivot in every column. Remember the only way for two vectors to be linearly dependent is for them to be multiples of each other. So the echelon form of A will be:

$$\begin{bmatrix} \square & \star \\ 0 & \square \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

26. A is a 4×3, $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

Since $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, both $\mathbf{a}_1$ and $\mathbf{a}_2$ are pivot columns in $[\mathbf{a}_1, \mathbf{a}_2]$, and hence in A. Since $\mathbf{a}_3$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$, the system $[\mathbf{a}_1, \mathbf{a}_2]\mathbf{x} = \mathbf{a}_3$ is inconsistent, so the augmented matrix for that system has a pivot in the last column. But the augmented matrix for that system is just A, so the last column of A, which is $\mathbf{a}_3$, is a pivot column as well. Thus A has a pivot in every column, and the echelon form is:

$$\begin{bmatrix} \square & \star & \star \\ 0 & \square & \star \\ 0 & 0 & \square \\ 0 & 0 & 0 \end{bmatrix}$$

27. How many pivot columns must a 7×5 matrix have if its columns are linearly independent?

Call the matrix A. Since the columns of A are linearly independent, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, so the system cannot have any free variables. This means that there is a pivot position in every column of the matrix A, so since there are 5 columns, there must be 5 pivots.

28. How many pivot columns must a 5×7 matrix have if its columns span $\mathbb{R}^5$?

Call the matrix A. Since the columns of A span $\mathbb{R}^5$, there must be a pivot position in every row of A. So since there are 5 rows, there must be 5 pivot positions.

29. Construct 3×2 matrices A and B such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

Since we want $A\mathbf{x} = \mathbf{0}$ to have only the trivial solution, we want the system not to have any free variables, which means we want matrix A to have a pivot position in every column. For example we could take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since we want $B\mathbf{x} = \mathbf{0}$ to have nontrivial solutions, we want the system to have free variables, which means we want B to have at least one column without a pivot position. For example we could take

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Any matrix A whose echelon forms is the following would work.

$$A = \begin{bmatrix} \square & \star \\ 0 & \square \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Any matrix B whose echelon form is one of the following would work.

$$\begin{bmatrix} 0 & \square \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \square & \star \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$