Newberger Math 247 Spring 03

Homework solutions: Section 1.4, #31-36

31. Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

Notice that saying the equation $A\mathbf{x} = \mathbf{b}$ *is consistent for all* $\mathbf{b}$ *in* $\mathbb{R}^3$ *is the same thing as saying that the columns of* A *span all of* $\mathbb{R}^3$*. Theorem 4 says that the columns of* A *span all of* $\mathbb{R}^3$ *only when* A *has a pivot position in every row. Since* A *is* 3×2 *there can be at most* 2 *pivots, so there cannot be a pivot in every row.*

Geometrically, we can see that the largest subset of $\mathbb{R}^3$ *that the* 2 *columns of* A *can span is a plane, so there will be some vectors* $\mathbf{b}$ *that will not lie in this plane, and the system* $A\mathbf{x} = \mathbf{b}$ *will not always be consistent.*

Generally, if A *has more rows than columns, there cannot be a pivot in every row, and the system* $A\mathbf{x} = \mathbf{b}$ *will not always be consistent.*

32. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when m is greater than n?

Let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ *be vectors in* $\mathbb{R}^4$*. Then the matrix* $[\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$ *can have at most* 3 *pivot positions, and hence cannot have a pivot in every row. So the vectors could not span all of* $\mathbb{R}^4$ *by Theorem 4. In general,* n *vectors in* $\mathbb{R}^m$ *cannot span all of* $\mathbb{R}^m$ *because the matrix with these column vectors can have at most* n *pivot positions, and will not have a pivot position in every row.*

33. Suppose that A is a 4×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of A? Justify your answer.

Since $A\mathbf{x} = \mathbf{b}$ *has a unique solution,* A *must have a pivot position in very column. In reduced echelon form, the matrix must have 0's above and below all the pivot positions and 1's in the pivot positions. So in reduced echelon form* A *must be*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

34. Suppose A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

Since $A\mathbf{x} = \mathbf{b}$ has a unique solution, there must be a pivot in every column of A. So since A is a 3×3 matrix, it has three columns, which means 3 pivot positions. But A also has 3 rows, so there is also a pivot in every row. This means that the columns of A span all of $\mathbb{R}^3$.

35. Let A be a 3×4 matrix and let $\mathbf{y}_1$ and $\mathbf{y}_2$ be vectors in $\mathbb{R}^3$, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$ for some vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent?

We can see that this is true since

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{y}_1 + \mathbf{y}_2 = \mathbf{w}.$$

So $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ is a solution to the system $A\mathbf{x} = \mathbf{w}$. Having a solution and being consistent are of course the same. The fact that we are looking for is $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$.

36. Let A be a 5×3 matrix, and let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and $\mathbf{z}$ a vector in $\mathbb{R}^5$. Suppose that $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{4z}$ is consistent?

Since $A\mathbf{y} = \mathbf{z}$, the vector $\mathbf{z}$ is a linear combination of the columns of A. Call the columns of A by $\mathbf{a_1}, \mathbf{a_2}$ and $\mathbf{a_3}$, and $\mathbf{y} = [y_1, y_2, y_3]$. We have

$$y_1\mathbf{a_1} + y_2\mathbf{a_2} + y_3\mathbf{a_3} = \mathbf{z}.$$

Multiply this equation by 4 on both sides. We see that $4z$ is also a linear combination f the columns of A, with weights $4y_1, 4y_2$ and $4y_3$. Thus $A(4\mathbf{y}) = \mathbf{4z}$, and the system $A\mathbf{x} = 4z$ is consistent, with solution $\mathbf{x} = 4\mathbf{y}$.

We can also see this through the matrix equation: multiply the equation $A\mathbf{y} = \mathbf{z}$ by 4 on both sides. We get $4A\mathbf{y} = \mathbf{4z}$, which by Theorem 5 becomes $A(4\mathbf{y}) = \mathbf{4z}$, hence the system $A\mathbf{x} = \mathbf{4z}$ has a solution. The fact we are looking for is $A(4\mathbf{y}) = 4A\mathbf{y}$.