Newberger Math 247 Spring 03

Exam 2 Solutions

1. a. (7 points) Calculate the determinant of the following matrix.

$$A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 \\ 1 & 7 & 2 & -5 \\ 8 & 3 & 1 & 8 \end{bmatrix}$$

I expanded on the second row of A.

$$\begin{aligned} \det A &= 2(-1)^{1+2} \det \begin{bmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{bmatrix} \\ &= (-2) \cdot 5(-1)^{1+3} \det \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \\ &= (-10)(7 \cdot 1 - 2 \cdot 3) \\ &= -10. \end{aligned}$$

b. (6 points) State the definition of one-to-one.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one if for every $\mathbf{b} \in \mathbb{R}^m$, there exists at most one $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

c. (6 points) State the definition of onto.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called onto if for every $\mathbf{b} \in \mathbb{R}^m$, there exists at least one $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

2. a. Let T be the transformation given by

$$T(x_1, x_2, x_3) = (-x_1^2, x_2 + x_3).$$

i. (2 points) What must a and b be so that $T : \mathbb{R}^a \to \mathbb{R}^b$?

$a = 3$ *and* $b = 2$.

ii. (6 points) Is T linear? Explain why or why not.

Your intuition should tell you this is not linear, since there is a square term in the first entry. However to show it is not linear, it is not enough to say that; you must show how this transformation fails to satisfy the definition of linear. First we check whether or not $T(\mathbf{0}) = \mathbf{0}$. If this were not true, then we would be done. Since it is truce we have to try something else.
To show T is not linear, we must exhibit either two vectors, $\mathbf{u}$ and $\mathbf{v}$, such that $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$, or a vector $\mathbf{u}$ and a scalar c such that $T(c\mathbf{u}) \neq cT(\mathbf{u})$. For example let $\mathbf{u} = (1, 1, 1)$ and let $c = -1$. Then

$$T(c\mathbf{u}) = T(-1, -1, -1) = (-1, -2),$$

while

$$cT(\mathbf{u}) = (-1)T(1, 1, 1) = (-1)(-1, 2) = (1, -2).$$

Since $(-1, -2) \neq (1, -2)$, we see that T is not linear.

b. Let T be the transformation given by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, -x_1 + x_2 + x_3).$$

i. (6 points) Is T linear? Explain why or why not.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 - x_3 \\ -x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus T is a matrix transformation, and since all matrix transformations are linear, T is linear.

ii. (6 points) Is T onto? Explain why or why not.

The standard matrix for T is

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix does not have a pivot in the last column, the system

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

will not be consistent for all $\mathbf{b}$ *in* $\mathbb{R}^2$. *So there are some* $\mathbf{b}$ *in* $\mathbb{R}^2$ *such that no* $\mathbf{x}$ *in* $\mathbb{R}^3$ *satisfy* $T(\mathbf{x}) = \mathbf{b}$, *and* T *does not satisfy the definition of onto.*

3. a. (7 points) Let A and B be 3×3 matrices. Suppose the third column of B is a linear combination of the first two columns of B. Explain why the third column of AB must be a linear combination of the first two columns of AB.

Let $B = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$. *We know that* $\mathbf{b}_3 = c\mathbf{b}_1 + d\mathbf{b}_2$, *where* c *and* d *are scalars. Now* $AB = [A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3]$, *so the third column of* AB *is* $A\mathbf{b}_3$. *Thus we have*

$$A\mathbf{b}_3 = A(c\mathbf{b}_1 + d\mathbf{b}_2) = A(c\mathbf{b}_1) + A(d\mathbf{b}_2) = cA\mathbf{b}_1 + dA\mathbf{b}_2.$$

Thus the third column of AB *is a linear combination of the first two columns of* AB.

b. (7 points) Let A, B, C and X be $n \times n$ matrices, such that A, B and C are invertible. Suppose $ABX = C$. Explain why X is invertible, and find X^{-1}.

Since A, B *and* C *are invertible, we have* A^{-1}, B^{-1} *and* C^{-1} *to work with.*

$$ABX = C$$

implies that

$$(A^{-1}A)BX = A^{-1}C,$$

so

$$BX = A^{-1}C.$$

Similarly we get

$$(B^{-1}B)X = B^{-1}A^{-1}C,$$

so

$$X = B^{-1}A^{-1}C.$$

Now we see that X *is the product of* A^{-1}, B^{-1} *and* C, *all of which are invertible (*A^{-1} *and* B^{-1} *are invertible since they are the inverses of invertible matrices). Thus* X *is invertible. Now taking the inverse of both sides, we get*

$$X^{-1} = (B^{-1}A^{-1}C)^{-1} = C^{-1}AB.$$

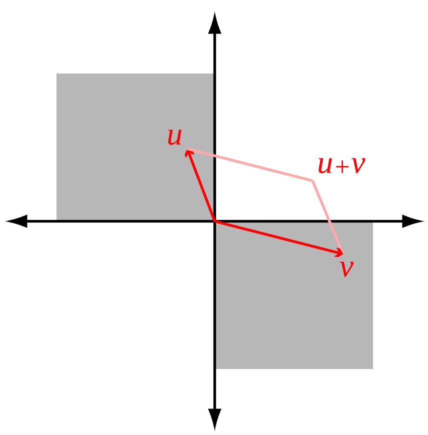
c. (7 points) Let A be an invertible $n \times n$ matrix. Is it possible for the equation $A\mathbf{x} = \mathbf{b}$ to be inconsistent for some $\mathbf{b}$ in $\mathbb{R}^n$? Explain why or why not.

Since A is invertible, the invertible matrix theorem says that A has a pivot position in every row, thus $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$. Thus it is not possible for $A\mathbf{x} = \mathbf{b}$ to be inconsistent for some $\mathbf{b}$ in $\mathbb{R}^n$.

4. a. (6 points) State the definition of a subspace of $\mathbb{R}^n$.

H is a subspace of a $\mathbb{R}^n$ if it satisfies the following:
- *i. The zero vector is in H.*
- *ii. H is closed under addition, i.e. if $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is in H.*
- *iii. H is closed under scalar multiplication, i.e. if c is a scalar and $\mathbf{u}$ is in H, then $c\mathbf{u}$ is in H.*

b. (7 points) Let W be the subset of $\mathbb{R}^2$ shown below, consisting of the second and fourth quadrants. Explain why W is not a subspace of $\mathbb{R}^2$ by saying (in words) which part or parts of the definition of a subspace fails to hold for W. Demonstrate this by drawing (and labelling) vectors on the graph.



This set is not closed under addition. The two vectors $\mathbf{u}$ and $\mathbf{v}$ are in W, but $\mathbf{u} + \mathbf{v}$ is not in W.

c. Fill in the blank (1 point each):
- i. If A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^k$ where $k = n$.

ii. If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^k$ where $k = m$.

iii. If A is an invertible $n \times n$ matrix, then Col $A = \mathbb{R}^n$.

iv. If A is an invertible $n \times n$ matrix, then Nul $A = \{\mathbf{0}\}$.

v. Suppose A is an $m \times n$ matrix. If $T(\mathbf{x}) = A\mathbf{x}$ is onto, then Col $A = \mathbb{R}^m$.

vi. Suppose A is an $m \times n$ matrix. If $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, then Nul $A = \{\mathbf{0}\}$.

5. a. (6 points) State the definition of a basis.

A basis of a subspace W is a set of vectors $\{v_1, \ldots, v_n\}$ such that $W =$Span$\{v_1, \ldots, v_n\}$ and such that $\{v_1, \ldots, v_n\}$ is a linearly independent set.

b. Let $A = \begin{bmatrix} 1 & -1 & -8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix}$.

i. (7 points) Find a basis for Col A.

$$A = \begin{bmatrix} 1 & -1 & -8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -8 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

So the first two columns of A are pivot columns. Thus a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

ii. (3 points) Give a geometric description of Col A, where A is the matrix given above.

Since Col A has 2 basis vectors, Col A is a plane passing through the basis vectors and zero, i.e. Col A is a plane through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

iii. (5 points) Do the columns of the matrix A given above form a basis for $\mathbb{R}^3$? Explain why or why not.

The columns of A do not form a basis for $\mathbb{R}^3$ since there is not a pivot in every row and column of A. Since A does not have a pivot in every row, the columns of A do not span $\mathbb{R}^3$, and since A does not have a pivot in every column, the columns of A are not linearly independent. Thus the definition of a basis for $\mathbb{R}^3$ is not met.