Newberger Math 247 Spring 03

Exam 1 name:________________________

1. The augmented matrix of a linear system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{array}\right].$$

a. Reduce the augmented matrix to reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

b. Write the general solution of the linear system in parametric vector form.

The general solution is

$$x_1 = -2 + x_3$$
$$x_2 = 3 - 2x_3$$
$$x_3 \text{ is free.}$$

So in parametric vector form, we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + x_3 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3,$$

where x_3 is any real number.

c. (6 points) Give a geometric description of the solution set.

The solution set is a line in $\mathbb{R}^3$ through $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ parallel to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Note that since there are three variables, the solution set lies in $\mathbb{R}^3$. If there had been 2 free variables, the geometric description would be a plane in $\mathbb{R}^3$.

2. The problems on this page refer to the following vectors in $\mathbb{R}^3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ h \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 3 & k & h \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & k & h-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & k-1 & h-7 \end{array}\right]$$

(a) (6 points) Find all values of h and k such that $\mathbf{b}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

In order for $\mathbf{b}$ *to be in Span*$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$*, the system must be consistent. This happens when* $k \neq 0$ *and* h *is any real number (in which case* $A\mathbf{x} = \mathbf{b}$ *has a unique solution) and when* $k = 0$ *and* $h = 0$ *(in which case* $A\mathbf{x} = \mathbf{b}$ *has infinitely many solutions).*

(b) Find all values of k such that the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span $\mathbb{R}^3$.

In order for the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ *to span* $\mathbb{R}^3$*, there must be a pivot position in every row. This happens when* $k \neq 1$*.*

(c) Find all values of k such that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly dependent set.

In order for $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ *to be a linearly dependent set, there must be a column without a pivot position (so that the homogeneous equation* $A\mathbf{x} = \mathbf{0}$ *has non-trivial solutions). This happens when* $k = 1$*.*

3. (a) Fill in the blank: The span of the columns of a 5×3 matrix lies in $\mathbb{R}^k$, where $k = 5$.

(b) Fill in the blank: The solution set of the homogeneous equation of a 3×2 matrix lies in $\mathbb{R}^k$, where $k = 2$.

The solution set will be in $\mathbb{R}^2$ since the matrix has two columns and the variables correspond to the columns. If you solved this

$$\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and wrote your solution in parametric vector form, there would be two entries in each of your solution vectors, so the solution set is in $\mathbb{R}^2$.

(c) If the augmented matrix of a linear system is 3×5, how many unknowns does the system have?

4 unknowns

(d) How many pivot positions must a 6×5 matrix have if the columns are linearly independent?

5 pivot positions

(e) How many pivot positions must a 5×3 matrix have if the solution set of the homogeneous equation is a line?

2 pivot positions
In this case, the system would have to have exactly one free variable, so it must have two pivot positions.

(f) How many pivot positions must a 3×5 matrix have if the columns span all of $\mathbb{R}^3$?

3 pivot positions

(g) Suppose the columns of a 2×2 matrix do not span $\mathbb{R}^2$. Describe all of the possible echelon forms of the matrix. Use the notation used in part (h).

This system can have at most one pivot position.

$$\begin{bmatrix} \blacksquare & \star \\ 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(h) Does the homogeneous equation of a matrix whose echelon form is given below have non-trivial solutions? How do you know?

$$\begin{bmatrix} \blacksquare & \star & \star & \star \\ 0 & 0 & \blacksquare & \star \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

where each ($\blacksquare$) is any non-zero real number and each ($\star$) is any real number.

In the homogeneous equation, x_2 is free since there is no pivot position in the second column. Thus the homogeneous equation has non-trivial solutions.

4. The problems on this page refer to the following vectors in $\mathbb{R}^3$.

$$\mathbf{u} = \begin{bmatrix} 4 \\ -12 \\ 8 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}.$$

(a) Are $\{\mathbf{u}, \mathbf{v}\}$ linearly independent? How do you know?

No. Since $\mathbf{u} = -\frac{4}{3}\mathbf{v}$*, the vectors are multiples of one another, and* $\{\mathbf{u}, \mathbf{v}\}$ *is linearly dependent.*

(b) Give a geometric description of Span$\{\mathbf{u}, \mathbf{v}\}$.

Since these vectors lie on the same line through the origin, Span$\{\mathbf{u}, \mathbf{v}\}$ *is a line through* $\mathbf{u}$ *and* $\mathbf{0}$.

(c) Name a vector that lies in Span$\{\mathbf{u}, \mathbf{v}\}$ but is neither $\mathbf{u}$ nor $\mathbf{v}$.

For example the vector $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ *lies in Span*$\{\mathbf{u}, \mathbf{v}\}$.

(d) (5 points) Name a vector in $\mathbb{R}^3$ that does not lie in Span$\{\mathbf{u}, \mathbf{v}\}$.

For example the vector $\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$ *does not lie in Span*$\{\mathbf{u}, \mathbf{v}\}$.

(e) Suppose A is a 3×2 matrix. Explain why the columns of A cannot span all of $\mathbb{R}^3$.

The columns of A *cannot span all of* $\mathbb{R}^3$ *because there are not enough possible pivot positions. To span all of* $\mathbb{R}^3$*, there must be a pivot position in every row, however* A *has 3 rows but only 2 possible pivot positions (since there can be at most one pivot position per column, and there are only 2 columns).*

5. (a) Suppose A is a 3×3 matrix and $\mathbf{c}$ is a vector in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{c}$ is inconsistent. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{z}$ has a unique solution? Explain your answer.

The augmented matrix for the system $A\mathbf{x} = \mathbf{c}$ must reduce to the form

$$\left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & b \end{array}\right]$$

where $b \neq 0$. So since there is a vector $\mathbf{c}$ in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{c}$ is inconsistent, A must have a row without a pivot. Thus A can have at most 2 pivot positions. But this means that there is at least one column without a pivot position, since there are 3 columns. So the system $A\mathbf{x} = \mathbf{z}$ if consistent will have a free variable and hence infinitely many solutions. Thus there is no $\mathbf{z}$ in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{z}$ has a unique solution.

(b) Let A be a 3×2 matrix with at least one non-zero entry. Suppose the columns of A are linearly dependent. Give a geometric description of the solution set of $A\mathbf{x} = \mathbf{0}$.

Since A has at least one non-zero entry, it must have at least one pivot position (the only way to have no pivot positions is for the matrix to be entirely filled with zeros). Since the columns of A are linearly dependent, there must be a column without a pivot. The augmented matrix for the system system $A\mathbf{x} = \mathbf{0}$ must reduce to the form

$$\left[\begin{array}{cc|c} \blacksquare & \star & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ or } \left[\begin{array}{cc|c} 0 & \blacksquare & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Combining these two pieces of information, we can see that A has one pivot position and one free variable. Thus the solution set of $A\mathbf{x} = \mathbf{0}$ is a line, and since the equation is homogeneous, the solution set passes through the origin.