Abstract Algebra, Newberger, Fall 2006
Tips and Remarks for Sections 6.2

p 145 #42 part (b): Prove that the set I of all matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $a, b \in \mathbb{R}$ is an ideal in the ring $T = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$.

If you use the definition of an ideal, make sure that you note that an ideal is a subring, so you must first prove that I is a subring. You may instead use Theorem 6.1, which provides a shortcut.

p 145 #42 part (c): Show that every coset in T/I can be written in the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I$.

This part is actually very short, but many people did not understand what the problem was asking, so I spend most of this explanation describing what the question asks.

By definition, $T/I = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I \middle| a, b \in \mathbb{R} \right\}$. This statement is asking you to show that in fact, $T/I = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I \middle| a \in \mathbb{R} \right\}$. You are being asked to show and equality of sets. It is clear that

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I \middle| a \in \mathbb{R} \right\} \subseteq \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I \middle| a, b \in \mathbb{R} \right\},$$

so you have to show that

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I \middle| a, b \in \mathbb{R} \right\} \subseteq \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I \middle| a \in \mathbb{R} \right\}.$$

Begin by letting $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I \in T/I$. Then to show that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I \in \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I \middle| a \in \mathbb{R} \right\},$$

you must show that there exists $c \in \mathbb{R}$ such that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + I.$$

This c that you are looking for is (obviously?) a. So what you really need to do is show that $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + I = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I$. Begin by proving that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mod I,$$

and then use Theorem 6.6 to conclude the result.

p 153 #30: Let T and I be as in Exercise #42 on page 145. Prove that $T/I \cong \mathbb{R}$.

You have two choices: you can prove it directly, using your the result from problem page 145 #42 part (c) that said $T/I = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I \middle| a \in \mathbb{R} \right\}$, or you can use the First Isomorphism Theorem. Here, I am going to direct you to solve the problem using the First Isomorphism Theorem.

To show that $T/I \cong \mathbb{R}$, using the First Isomorphism Theorem, you must construct a function $\phi : T \to \mathbb{R}$ such that ϕ is surjective, a homomorphism and such that $\ker \phi = I$.

Here is how I knew which function to take. Consider $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$. We want $I = \ker \phi$, so we want to have $\phi\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = 0$.

The domain of ϕ is T, so we need to be able to plug any matrix of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ into ϕ. Since we want the answer to be zero precisely when $a = 0$, it looks like taking

$$\phi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = a$$

will give us the right kernel.

Indeed, this function will work; begin your proof by stating what ϕ is. Then, you should prove that the ϕ given above is (1) surjective, (2) a homomorphism, and that (3) $\ker \phi = I$. To do (3), note that

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$$

and

$$\ker \phi = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in T \middle| \phi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = 0 \right\}.$$

Prove that $I \subseteq \ker \phi$ and that $I \supset \ker \phi$.

p 152 #18 Let $I \neq R$ be an ideal in a commutative ring R with identity. Prove that R/I is an integral domain if and only if whenever $ab \in I$, either $a \in I$ or $b \in I$.

This argument will use our Lemma (*): $a \in I \Leftrightarrow a + I = 0_R + I$. Prove this statement in your homework (it is in your class notes, if you need a hint).

"$\Rightarrow$" Let R/I be an integral domain. Consider the definition of integral domain: let S be an integral domain, and let $x, y \in S$. By definition, we have if $xy = 0_S$. then $x = 0_S$ or $y = 0_S$. Now in this problem, S is R/I, and the elements of R/I are cosets, with the zero element being $O_R + I$. In your proof, begin by explaining what the definition of integral domain says in this case, using cosets $a + I$ and $b + I$ instead of x and y.

We want to show that if $ab \in I$, then $a \in I$ or $b \in I$. What we want to show is an if-then statement, so we begin its proof by assuming the if part. Let $ab \in I$.

Now use Lemma (*) to change this into a statement about cosets. Remember that $ab+I = (a+I)(b+I)$. Now use the definition of integral domain that you explained above to conclude that either $a+I = 0_R + I$ or $b+I = 0_R + I$, and use Lemma (*) to get the desired result.

"$\Leftarrow$" Suppose that for any $a, b \in R$, if $ab \in I$ then $a \in I$ or $b \in I$. We want to show that R/I is an integral domain. Begin by explaining that R/I is a commutative ring with identity, using Theorem 6.9. Now note that since $I \neq R$, the identity $1_R \notin I$ (this is Exercise 13, page 142). This means, by Lemma (*), that $1_R + I \neq 0_R + I$, as required by the definition of integral domain.

Now explain what you want to show, using the definition of integral domain that you explained in the proof of "$\Rightarrow$." Since what you want to show is an if-then statement, begin by assuming the if part: suppose that $(a+I)(b+I) = 0_R + I$. Use Lemma (*) (twice) and the assumptions in this part to show that $a+I = 0_R+I$ or $b+I = 0_R+I$.