**Abstract Algebra, Newberger, Fall 2006**

**Homework: Sections 4.3**
*due Thursday, October 19*

**II. Page 98 #8** Let $F$ be a field and let $f(x)$ be a nonzero polynomial in $F[x]$. If $f(x)$ can be written as the product of two polynomials of lower degree, then $f(x)$ is reducible in $F[x]$.

Write down your assumptions: Suppose that $f(x) = r(x)s(x)$, where $deg(r(x)) < deg(f(x))$ and $deg(s(x)) < deg(f(x))$.

Write down what you want to show: We want to show that $f(x)$ is reducible. This means that we want to show that $f(x)$ is not irreducible. A polynomial is irreducible if every divisor is either a unit or an associate. Thus we want to show that $f(x)$ has a divisor that is neither a unit nor an associate.

In other words, our goal is to find a divisor of $f(x)$ and prove that that divisor is not an associate and is not a unit. A good candidate for the divisor we are searching for is $r(x)$ (or $s(x)$, of course; pick one).

So we have $f(x) = r(x)s(x)$, with $deg(r(x)) < deg(f(x))$ and $deg(s(x)) < deg(f(x))$, and we want to show that $r(x)$ is not an associate of $f(x)$ and not a unit. Proceed by taking the degree of both sides of the equation $f(x) = r(x)s(x)$. Argue by contradiction: suppose that $r(x)$ is constant, and find a contradiction, and then suppose that $r(x)$ is an associate and get a contradiction.

**III. Page 99 #16** Prove that $p(x)$ is irreducible in $F[x]$ if and only if for every $g(x) \in F[x]$, either $p(x)|g(x)$ or $p(x)$ is relatively prime to $g(x)$.

"$\Rightarrow$"

Write down your assumptions: Suppose that $p(x)$ is irreducible.

Write down what you want to show: We want to show that for every $g(x) \in F[x]$, either $p(x)|g(x)$ or $p(x)$ is relatively prime to $g(x)$. To prove a statement about every element of $F[x]$, choose an arbitrary element of $F[x]$, and prove it for that one. Then conclude that since your element was arbitrary, your proof would hold for any other element of $F[x]$ as well, so you have, in fact, proved it for every element of $F[x]$. So begin by saying, Let $g(x) \in F[x]$ be arbitrary.

Now you want to show that either $p(x)|g(x)$ or $p(x)$ and $g(x)$ are relatively prime. This proof is similar to the proof of 4.11 (1)$\Rightarrow$(2). Consider the greatest common divisor $d(x)$ of $p(x)$ and $g(x)$, and use the irriduciblity of $p(x)$ to show that either $p(x)|g(x)$ or $d(x) = 1$ (and hence $p(x)$ is relatively prime to $g(x)$.

"$\Leftarrow$"

Write down your assumptions: Suppose that for every $g(x) \in F[x]$, either $p(x)|g(x)$ or $p(x)$ is relatively prime to $g(x)$. This statement is a statement about every element of $F[x]$. To apply it you need a particular element of $F[x]$ to begin with.

Write down what you want to prove: We want to show that $p(x)$ is irreducible. To do this, let $h(x)$ be a divisor of $p(x)$ and prove that $h(x)$ is either an associate of $p(x)$ or a unit.

Now you have a particular element of $F[x]$ to which you can apply your assumptions. I.e. the assumptions are true for every element of $F[x]$, so they will be true for $h(x)$. So either $p(x)|h(x)$ or $p(x)$ and $h(x)$ are relatively prime.

Thus we have the assumption $h(x)|p(x)$, and we will prove the statement in two cases: (1) $p(x)|h(x)$ and (2) $p(x)$ and $h(x)$ are relatively prime. In case (1), prove that $h(x)$ is an associate of $p(x)$. In case (2), prove that $h(x)$ is constant.