Abstract Algebra, Newberger, Fall 2006

Tips and Remarks for the Homework from Section 3.2

I. Uniqueness. Page 62 #3.

(a) Can a ring R have more than one zero element?
The answer is no. There are a variety of ways to prove this; I will suggest you follow the hint, as follows.

To prove that something is unique, assume that there are two of them, and then prove that they are equal to each other. In this case, we have one additive identity, 0_R, so our assumption is that there is a second additive identity z. Now use Theorem 3.3; explain that both z and 0_R must be solutions to the equation $0_R + x = 0_R$, and conclude that they must be equal.

(b) Can a ring with identity have more than one identity element?
Again the answer is no. Prove this similarly, using Theorem 3.8.

II. Subrings. Page 63 #7 (c).

(c) Use Theorem 3.6.

III. Zero divisors. Fundamental: page 63#16

(a) If ab is a zero divisor in a ring R, prove that a or b is a zero divisor.

Begin by writing your assumptions (the if part). Suppose ab is a zero divisor. Then ab does not equal zero, and there exists $c \in R$ such that $abc = 0_R$ or $cab = 0_R$ (see the definition of a zero divisor, page 62). I will guide you to prove the statement in the case that $abc = 0_R$; your solution should also include a proof in the case that $cab = 0_R$. Note that the fact that $ab \neq 0_R$ implies that $a \neq 0_R$ and $b \neq 0_R$ (since if either a or b were zero, ab would also be 0_R).

Case 1: Suppose that there exists $c \in R$ such that $abc = 0_R$.

Now we write down what we want to prove: We want to show that either a or b is a zero divisor. Here is a standard approach to proving an or statement: either a is a zero divisor or a is not a zero divisor. If a is a zero divisor, we are done, so suppose that a is not a zero divisor. Then use that fact together with $a(bc) = 0_R$ to argue that $bc = 0_R$. Verify that b satisfies the definition of a zero divisor. The general approach to proving an or statement is to assume one of the statements (a is a zero divisor) is false, and prove that the other statement (b is a zero divisor) is true.

Case 2: Suppose that there exists $c \in R$ such that $cab = 0_R$. I leave this case to you.

IV. Units. page 65 #28.
Let R be a ring with identity. Prove that if ab and a are units in R, then b is a unit.

Begin by writing your assumptions (the if part). Suppose that ab and a are units in R. Then there exists $c, d \in R$ such that $abc = 1_R$, $cab = 1_R$, $ad = 1_R$ and $da = 1_R$. Manipulate these equations (by multiplying both sides by well chosen elements of R and using substitution) to find an element $e \in R$ such that $be = 1_R$. Then check that the element e that you found also satisfies $eb = 1_R$, and conclude that b is a unit.