Math 561B Newberger Spring 04
Take Home Midterm Exam

Please work independently on these problems; do not collaborate with other students or professors.

Let $0 < p \leq \infty$, and let p' satisfy $\frac{1}{p}+\frac{1}{p'} = 1$. A sequence $(f_k) \subset L^p(E)$ is said to converge weakly to a function f in $L^p(E)$ if for all $g \in L^{p'}$, the sequence of numbers $\int_E f_k g$ converges to $\int_E fg$.

In these problems, you will discuss the relationship between weak convergence in L^p and convergence in L^p. Part (1) says that convergence in L^p implies weak convergence in L^p. Part (2) says that weak convergence in L^p does not imply convergence in L^p (in particular you will consider $p = 2$). Part (3) says that weak convergence in L^p together with an additional assumption does imply convergence in L^p (again you will only consider the case $p = 2$).

(1) Let $1 \leq p \leq \infty$. Let $(f_k) \subset L^p(E)$ be a sequence. Prove that if (f_k) converges to f in $L^p(E)$, then (f_k) converges weakly to f.

(2) (a) Prove that if $g \in L^2(E)$, then

$$\lim_{k\to\infty} \int_0^{2\pi} g(x) \cos kx \, dx = \lim_{k\to\infty} \int_0^{2\pi} g(x) \sin kx \, dx = 0.$$

(b) Prove that there exists a sequence $(f_k) \subset L^2(E)$ such that (f_k) converges weakly to f, but (f_k) does not converge to f in $L^2(E)$. (Use part (a).)

(3) Let $(f_k) \subset L^2(E)$ be a sequence, and let $f \in L^2(E)$. Suppose that (f_k) converges weakly to f in $L^2(E)$. Suppose also that the sequence of numbers $\|f_k\|_2$ converges to $|f\|_2$. Prove that (f_k) converges to f in $L^2(E)$.

You may want to use a table of integrals to calculate any trigonometric integrals that come up in (2).