Section 5.1 #12 *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous on* $\mathbb{R}$ *and that* $f(r) = 0$ *for every* $r \in \mathbb{Q}$*. Prove that* $f(x) = 0$ *for every* $x \in \mathbb{R}$*.*

We want to prove that $f(x) = 0$ for every $x \in \mathbb{R}$. This is a "for every" statement; we will begin by fixing $x \in \mathbb{R}$. In your proof start with "Let $x \in \mathbb{R}$." Now we will show that $f(x) = 0$. Remember that x is a particular fixed number.

To show that $f(x) = 0$, we will show that $|f(x)| < \varepsilon$ for every $\varepsilon > 0$. Explain this in your write-up. Since this too is a for every statement, begin its proof by "Let $\varepsilon > 0$."

Now explain that since f is continuous on $\mathbb{R}$, it is continuous at the point x that we are studying. This means that for the ε that you fixed above, there exists a $\delta > 0$ that will ensure the following statement is true: if y is any number satisfying $|x - y| < \delta$, then y will also satisfy $|f(x) - f(y)| < \varepsilon$.

Prove that there exists a rational number r satisfying $|x - r| < \delta$. In other words, prove that there exists a rational number r such that $x - \delta < r < x + \delta$. Use the density property of the rational numbers to do this.

Now since $|x - r| < \delta$, r will also satisfy $|f(x) - f(r)| < \varepsilon$, by our choice of δ. Complete the proof from here, using the remaining assumption about f.