Section 4.1 #12 *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $\lim_{x\to 0} f(x) = L$ *and let* $a > 0$. *If* $g : \mathbb{R} \to \mathbb{R}$ *is given by* $g(x) = f(ax)$, *then prove* $\underset{x\to 0}{g}(x) = L$.

How to figure it out.

What do we want to prove? We want $\lim_{x\to 0} g(x) = L$, which is a "for every" statement. Begin with "Let $\varepsilon > 0$." Find a δ such that if $|x - 0| = |x| < \delta$, then $|g(x) - L| < \varepsilon$. So we want to find a $\delta > 0$ such that if $|x| < \delta$, then $|f(ax) - L| < \varepsilon$.

Now we know we are looking for a δ. What do we know? We are given that $\lim_{x\to 0} f(x) = L$, so for the ε that we chose, we can find a δ_o such that if $|x| < \delta_o$, then $|f(x) - L| < \varepsilon$. Now δ_o has now been chosen so that *any number* y satisfying $|y| < \delta_o$ will be guarantied to satisfy $|f(y) - L| < \varepsilon$. For example, since $\delta_o/2 < \delta_o$, we know $|f(\delta_o) - L| < \varepsilon$, though that is not very useful, and should not appear in your proof!

This means that to make $|f(ax) - L| < \varepsilon$, all you have to do is make $|ax| < \delta_o$. Choose δ so that if $|x| < \delta$, then $|ax| < \delta_o$, from which you can immediately conclude that $|f(ax) - L| < \varepsilon$, as desired.

What to write.

Let $\varepsilon > 0$. Then since $\lim_{x\to 0} f(x) = L$, there exists δ_o such that if $|x| < \delta_o$, then $|f(x) - L| < \varepsilon$. Now let $\delta =$(what?). Then suppose $|x| < \delta$ and prove that $|g(x) - L| < \varepsilon$.