Newberger Analysis 361A Fall 03

Homework Worksheet for Example 3.7.6(c) and (d)

In this assignment you will prove that $\sum \frac{1}{n^p}$ converges when $p > 1$. This is in the text Examples 3.7.6(c) and (d), so you should use the book for reference. The following steps will lead you to write out all the steps for the proof, and understand the argument in the book.

(1) This argument uses Theorem 3.7.5. State this theorem in your homework. Explain that this theorem says that to complete the proof, you need to prove that the partial sums $s_k = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$ are bounded.

(2) Prove the following statement:

> Suppose (x_k) is increasing and non-negative. Suppose (x_{k_m}) is a subsequence of (x_k) and that (x_{k_m}) is bounded ((x_{k_m}) is bounded means there exists an M such that $|x_{k_m}| \leq M$ for all $m \in \mathbb{N}$). Prove that (x_k) is bounded.

Apply this statement to (s_k), and conclude that to prove s_k converges, you need to find a bounded subsequence (s_{k_m}) of (s_k). Explain this in your homework.

(3) Let $k_m = 2^m - 1$. Write out the partial sums s_{k_1}, s_{k_2}, s_{k_3} and s_{k_4}. Show that

$$\begin{aligned} s_{k_1} &= 1, \\ s_{k_2} &< s_{k_1} + \frac{1}{2^{p-1}}, \\ s_{k_3} &< s_{k_2} + \frac{1}{(2^{p-1})^2}, \\ s_{k_4} &< s_{k_3} + \frac{1}{(2^{p-1})^3}. \end{aligned}$$

Conclude that

$$s_{k_m} < s_{k_{m-1}} + \frac{1}{(2^{p-1})^{m-1}} < 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots + \frac{1}{(2^{p-1})^{m-1}}.$$

For 5 points of extra credit, prove $s_{k_m} < s_{k_{m-1}} + \frac{1}{(2^{p-1})^{m-1}}$ by induction.

(4) Prove

$$1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots + \frac{1}{(2^{p-1})^{m-1}} < \frac{1}{1 - \frac{1}{2^{p-1}}},$$

for every $m \in \mathbb{N}$. Use this to explain that (s_{k_m}) is a bounded subsequence of (s_k).

(5) Conclude by explaining again why this completes the proof.