Newberger Analysis 361A Fall 2003

**Section 2.1 #18.**

Let $a, b \in \mathbb{R}$ and suppose that for every $\epsilon > 0$, we have $a \leq b + \epsilon$. Show that $a \leq b$.

Like the proof of Theorem 2.1.9, this statement will be easier to prove in the contrapositive. The original statement as an "if-then" is:

> *If (for every $\epsilon > 0$, we have $a \leq b + \epsilon$), then $a \leq b$.*

The contrapositive of this statement reads:

> *If ($a \leq b$) is not true, then (for every $\epsilon > 0$, we have $a \leq b + \epsilon$) is not true.*

If *(for every $\epsilon > 0$, we have $a \leq b + \epsilon$)* is not true, then not all $\epsilon > 0$ satisfy $a \leq b + \epsilon$. (Right? This is the key point here!) So our statement becomes:

> *If ($a \leq b$) is not true, then there is an $\epsilon > 0$ for which ($a \leq b + \epsilon$) is not true.*

Alternatively:

> *If $a > b$, then there is an $\epsilon > 0$ for which $a > b + \epsilon$.*

This is what you should prove. I suggest you read this over until it makes perfect sense to you, and then you explain it back to me on your homework, without looking at this slip. To prove it, set it up this way: "Assume that $a > b$. We want to show that there exists a number $\epsilon > 0$ such that $a > b + \epsilon$." It is often easier to know what to do if you change the expression "we want to show there exists" to "we want to find," since it gives you something to look for.

Here's a hint on how to find that $\epsilon > 0$. On a piece of scratch paper, pick out two numbers $a, b \in \mathbb{R}$ with $a > b$. Take for example $a = 8$ and $b = 6$. Can you find a number $\epsilon > 0$ such that $a > b + \epsilon$ for your example? How does that number depend on $a$ and $b$? To prove the statement, you should find a formula for a number $\epsilon$ in terms of $a$ and $b$ that satisfies $a > b + \epsilon$. Verify that the $\epsilon$ that you find is greater than zero.