Newberger Math 361A Spring 03

Section 1.1 #14 Show that the function f defined by $f(x) := x/\sqrt{x^2+1}$, $x \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.

Guidance:
When you write the solution to a mathematics problem, it is a good rule of thumb to "Say what you are going to do, do it, then say what you did." In this problem, you should do this for the overall problem, which involves 1) explaining that to prove f is a bijection, you must prove f is one-to-one and onto, then 2) doing the proofs, then 3) concluding at the end that f is a bijection. You should also do this inside the proofs that f is one-to-one and onto.

Proving f is one-to-one

- ***Explain what you are going to do.*** To prove f is one-to-one, let x_1 and x_2 be elements of the domain of f (be specific: say what the domain of f is in this problem). Say that you are going to prove that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- ***Do it.*** The following outline is designed to help you maximize your potential to figure out how to prove a statement correctly and efficiently. Do not number these steps in the argument, since you do not want your argument to look like a list.
  (1) *Write down what you know.* This means write down your assumptions, and explain what they mean if they involve vocabulary. In this problem, begin with something like "Suppose $f(x_1) = f(x_2)$," or "We assume that $f(x_1) = f(x_2)$." Explain what this means by saying that this implies $\frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}}$.
  (2) *Explain what you want to prove.* If you know what to do after you "Write what you know," skip this step. If you are confused, or do not know what to do next, this step might help. Say something like, "We want to prove that $x_1 = x_2$." The point of this step is to make sure you know where you are going with your argument.
  (3) *Begin with what you know, and explain how this leads to what you want to prove.* In this case, this means trying to simplify the expression $\frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}}$ to obtain $x_1 = x_2$.
- ***Explain what you did.*** At this point, you just need to acknowledge that you know you are finished. Say something like "Thus $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, and f is one-to-one."

Tip: If you have $x_1^2 = x_2^2$ at some point in your argument, you can conclude that this means $x_1 = x_2$ or $x_1 = -x_2$. Do not forget that there are both positive and negative solutions!

Tip: A list of equations is not an argument. Put the words "which implies" or "which means" or the symbol $\Longrightarrow$ (which means "which implies") between consecutive equivalent equations. For example:

$$\begin{aligned} & f(x_1) = f(x_2) \\ \Longrightarrow \quad & \frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}} \\ \Longrightarrow \quad & x_2\sqrt{x_1^2+1} = x_1\sqrt{x_2^2+1}. \end{aligned}$$

Proving f is onto

- ***Explain what you are going to do.*** To prove f is onto, let y be an element of the codomain of f (be specific: say what the codomain of f is in this problem). Say that you are going to prove that there is at least one x in the domain of f (be specific: say what the domain of f is in this problem) such that $f(x) = y$.

- ***Do it.*** The following outline is designed to help you maximize your potential to figure out how to prove a statement correctly and efficiently. Do not number these steps in the argument, since you do not want your argument to look like a list.
  (1) *Write down what you know.* This means write down your assumptions, and explain what they mean if they involve vocabulary. In this problem, your only assumption is that y is in the interval $(-1, 1)$, and that $f(x) = \frac{x}{\sqrt{x^2+1}}$.
  (2) *Explain what you want to prove.* Say something like, "We want to find an x in the domain of f such that $f(x) = y$." (Always be specific to the problem: say what the domain of f is in this case.) Explain what this means. For example, you could say something like: "This means that we want to find x in the domain of f such that $\frac{x}{\sqrt{x^2+1}} = y$. To do this, we solve for x." At this point, go ahead and solve for x. It is as important to explain why solving for x is the right thing to do as it is to actually make the calculation! Read the Tips after the one-to-one discussion above before writing this part.
  (3) *Begin with what you know, and explain how this leads to what you want to prove.* Now explain why knowing that y is in the codomain of f (again, be specific: say what the codomain of f is in this case) implies that the x that you found is in the domain of f (still being specific), and show that when you evaluate the function f at the x that you found, you actually do get $f(x) = y$.

- ***Explain what you did.*** At this point, you just need to acknowledge that you know you are finished. Say something like "Thus we have shown that for every y with $-1 < y < 1$, there exists an $x \in \mathbb{R}$, namely $x =$your answer, such that $f(x) = y$, and hence f is onto."