Newberger Analysis 361A Fall 03

Exam 2 name:________________________

1.________

2.________

3.________

1. (a) (15 points) Prove the Comparison Test: Let (x_n) and (y_n) be real sequences and suppose that for some $K \in \mathbb{N}$, we have

$$0 \leq x_n \leq y_n \text{ for } n \geq K.$$

Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.

(b) (7 points) State the definition of a convergent series.

(c) (7 points) State the definition of the limit of a function.

(d) (7 points) State the definition of continuous (at a point).

2. (a) (14 points) Let $A \subseteq \mathbb{R}$ and let c be a cluster point for A. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, and let $h = f + g$. Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c} h(x) = L + M$.

Proof. We want to prove that $\lim_{x\to c} h(x) = L + M$, so we want to prove that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$, then $|h(x) - h(c)| < \varepsilon$.

Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there exists a $\delta_1 > 0$ such that if $|x - c| < \delta_1$, then $|f(x) - L| < \varepsilon/2$. Since $\lim_{x\to c} g(x) = M$, there exists a $\delta_2 > 0$ such that if $|x - c| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Now we will prove if $|x - c| < \delta$, then $|h(x) - h(c)| < \varepsilon$. Suppose $|x - c| < \delta$. Then $|x - c| < \delta \leq \delta_1$ and $|x - c| < \delta \leq \delta_2$. Thus by our choices of δ_1 and δ_2 above, $|f(x) - L| < \varepsilon/2$ and $|g(x) - M| < \varepsilon/2$. From this we have

$$\begin{aligned}|h(x) - (M + L)| &= |f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \\ &\leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

as desired. □

(b) Prove that the following functions are not continuous at $c = 2$.

(i) (8 points) $g(x) = \dfrac{1}{x - 2}$.

This function is not defined at $x = 2$, so it is not continuous at 2. You can prove that $\lim_{x\to 2} \frac{1}{x-2}$ does not exist by considering the sequence given by $x_n = 2 + \frac{1}{n}$. Note that (x_n) converges to 2, but that $g(x_n) = n$ diverges.

(ii) (8 points) $f(x) = \begin{cases} x & \text{if } x \geq 2 \\ -x & \text{if } x < 2 \end{cases}$.

We will show this function is not continuous at 2 by finding two sequences (x_n) and (y_n) converging to 2, but such that $\lim\,(f(x_n)) \neq \lim\,(f(y_n))$. Let $x_n = 2 + \frac{1}{n}$, and let $y_n = 2 - \frac{1}{n}$. Now $f(x_n) = f(2 + \frac{1}{n}) = 2 + \frac{1}{n}$ since $2 + \frac{1}{n}$ is greater than 2, and $f(y_n) = f(2 - \frac{1}{n}) = -(2 - \frac{1}{n})$ since $2 - \frac{1}{n}$ is less than 2. Thus $\lim\ f(x_n) = 2$, while $\lim f(y_n) = -2$. So f is not continuous at 2.

3. (a) Determine whether or not the following series are convergent. Prove your answer.

(i) (8 points) $\sum \frac{1}{n^2-2}$.

Let $x_n = \frac{1}{n^2}$ and let $y_n = \frac{1}{n^2-2}$. Then $\lim \frac{x_n}{y_n} = \lim \frac{n^2}{n^2-2} = 1$. By the limit comparison test, since $\sum \frac{1}{n^2}$ converges, we have that $\sum \frac{1}{n^2-2}$ converges.

(ii) (8 points) $\sum 2^n$.

By the n^{th}-term test, since $\lim 2^n$ does not equal zero (in fact it does not exist at all), we have that $\sum 2^n$ diverges. Alternatively, all geometric series of the form $\sum r^n$ with $|r| > 1$ diverges, so since this series is geometric series with $r = 2$, it diverges.

(b) (i) (6 points) What do you have to prove to show that a sequence is not Cauchy, using the definition?

To prove a sequence (x_n) is not Cauchy, I must show that there exists a $\varepsilon_o > 0$ such that for every $H \in \mathbb{N}$, there are numbers $m, n \geq H$, with $|x_m - x_n| \geq \varepsilon_o$.

(ii) (8 points) Prove the sequence $(x_n) = \left(\left(1 - \frac{1}{n}\right)(-1)^n\right)$ is not Cauchy, using the definition.

Let $\varepsilon_o = 1$. Let $H \in \mathbb{N}$. Let $m = 2H$ and $n = 2H+1$. Note that since $H \in \mathbb{N}$, we have $2H \geq 2$, and the largest that $\frac{1}{2H}$ can be is $\frac{1}{2}$, while the largest that $\frac{1}{2H+1}$ can be is $\frac{1}{3}$. Then we get

$$\begin{aligned} |x_n - x_m| &= \left|\left(1 - \frac{1}{2H}\right)(-1)^{2H} - \left(1 - \frac{1}{2H+1}\right)(-1)^{2H+1}\right| \\ &= \left|1 - \frac{1}{2H} + 1 - \frac{1}{2H+1}\right| \\ &= \left|2 - \frac{1}{2H} - \frac{1}{2H+1}\right| \geq 2 - \frac{1}{2} - \frac{1}{3} > 1. \end{aligned}$$