Newberger Analysis 361A Fall 03

Exam 2 Solutions

1. (a) Prove that $\lim \dfrac{4n}{2n-1} = 2$.

Proof. Let $\varepsilon > 0$. Let $K > \dfrac{2+\varepsilon}{2\varepsilon}$. We will prove if $n \geq K$, then $\left|\dfrac{4n}{2n-1} - 2\right| < \varepsilon$. Let $n \geq K$. Then $n > \dfrac{2+\varepsilon}{2\varepsilon}$ so $2\varepsilon n - \varepsilon > 2$ and we have $\varepsilon > \dfrac{2}{2n-1}$. Note that

$$\left|\frac{4n}{2n-1} - 2\right| = \left|\frac{4n-4n+2}{2n-1}\right| = \frac{2}{2n-1}.$$

So for $n \geq K$ we get $\left|\dfrac{4n}{2n-1} - 2\right| < \varepsilon$ as desired. □

(b) Let (x_n) be a sequence with $\lim(x_n) = x$, and suppose that $x_n \neq 0$ for all $n \in \mathbb{N}$, and that $x \neq 0$. Prove that $\left(\dfrac{1}{x_n}\right)$ converges to $\dfrac{1}{x}$.

Proof. Let (x_n) be a sequence with $\lim(x_n) = x$, and suppose that $x_n \neq 0$ for all $n \in \mathbb{N}$, and that $x \neq 0$. Let $\varepsilon > 0$. Since $M > 0$, $x > 0$ and $\varepsilon > 0$, we have $Mx\varepsilon > 0$. Since $\lim(x_n) = x$, there exists a K such that if $n \geq K$, then $|x_n - x| < Mx\varepsilon$.
Now since $x_n > M$, we have $\dfrac{1}{x_n} < \dfrac{1}{M}$, so $\dfrac{1}{x_n x} < \dfrac{1}{Mx}$.
We will show if $n \geq K$, then $\left|\dfrac{1}{x_n} - \dfrac{1}{x}\right| < \varepsilon$. Let $n \geq K$. We have

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x_n x}\right| < \frac{|x - x_n|}{Mx} < \frac{Mx\varepsilon}{Mx} = \varepsilon,$$

as desired. □

2. (a) State the definition of the limit of a sequence.
   (b) State the definition of a bounded sequence.
   (c) Prove the following statement from Theorem 3.2.3(a): Let (x_n) and (y_n) be sequences of real numbers with $\lim x_n = x$ and $\lim y_n = y$. Prove that $(x_n y_n)$ converges to xy.

3. Give an example that shows that each of the following statements are ***false***.
   (a) *If (x_n) converges and $x_n > M$ for all n, then $\lim(x_n) > M$.*

   Let $x_n = \frac{1}{n}$, and let $M = 0$. Then $x_n > M$ for all $n \in \mathbb{N}$, but $\lim(x_n) = M$, so $\lim(x_n)$ is not strictly greater than M.

   (b) *Every subsequence of an unbounded sequence is unbounded.*

   Let $x_n = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ n & \text{if } n \text{ is even} \end{cases}$. Then the sequence (x_n) is unbounded, but the subsequence (x_{2n+1}) is constant and equal to zero, and hence bounded.

   (c) *Every bounded sequence converges.*

   Let $x_n = (-1)^n$. Then (x_n) does not converge since it has two subsequences, namely (x_{2n}) and (x_{2n+1}) that converge to different limits ((x_{2n}) converges to 1 and (x_{2n+1}) converges to -1).

   (d) *Every convergent sequence is monotone.*

   Let $x_n = \dfrac{(-1)^n}{n}$. Then x_n is not monotone, but (x_n) converges to 0.

   (e) *The sum of two monotone sequences is monotone.*

   Let (x_n) be the sequence $(1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ...)$ and let (y_n) be the sequence $(-1, -2, -2, -3, -3, -4, -4, -5, -5, ...)$. Then (x_n) is increasing and (y_n) is decreasing $(x_n + y_n)$ is the sequence $(0, -1, 0, -1, 0, -1, 0, ...)$, which is not monotone.