

1. (a) (8 points) State the definition of an upper bound for a set.  
 (b) (8 points) State the definition of the supremum of a set.  
 (c) (7 points) What is the supremum of the set  $S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$ ?

**Answer:**  $\sup S = 1$ .

- (d) (7 points) What is the infimum of the set  $S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$ ?

**Answer:**  $\inf S = \frac{1}{2}$ .

- (e) (8 points) Let  $S \subseteq \mathbb{R}$ , and let  $u = \sup S$ . Suppose  $v < u$ , and explain why there exists  $s \in S$  such that  $s > v$ .

**Solution:**

Since  $u = \sup S$ , if  $v$  is any upper bound for  $S$  then  $v \geq u$ . So since we are given that  $v < u$ , we know that  $v$  is **not** an upper bound for  $S$ . An upper bound is a number that is greater than or equal to every element of  $S$ . Since  $v$  is not an upper bound, there must be elements of  $S$  larger than  $v$ . i.e. There exists  $s \in S$  such that  $s > v$ .

- (f) (4 points—*This part is not worth much. Only do it if you have time remaining*)  
 Prove that your answer to part (c) is correct.

**Solution:** First we prove that 1 is an upper bound for  $S$ . Elements of  $S$  are of the form  $\frac{n}{n+1}$ . Since  $n+1 > n$ , we have  $\frac{n+1}{n+1} > \frac{n}{n+1}$  so  $1 > \frac{n}{n+1}$ . Thus 1 is an upper bound for  $S$ .

Second we will show that for every  $\varepsilon > 0$ , there exists  $s \in S$  such that  $s > 1 - \varepsilon$ , i.e. for every  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{n}{n+1} > 1 - \varepsilon$ . This is a “for every-there exists” problem, so we begin with:

Let  $\varepsilon > 0$ . Note that  $\frac{n}{n+1} = 1 - \frac{1}{n+1}$ . So we want to find  $n \in \mathbb{N}$  such that  $1 - \varepsilon < 1 - \frac{1}{n+1}$ . Let  $n \in \mathbb{N}$  be any natural number bigger than  $\frac{1}{\varepsilon} - 1$ , i.e. such that

$$n > \frac{1}{\varepsilon} - 1.$$

Then

$$\varepsilon > \frac{1}{n+1},$$

which implies

$$-\varepsilon < -\frac{1}{n+1},$$

and so

$$1 - \varepsilon < 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

as desired.

2. Let  $b \in \mathbb{R}$ . Consider the following statement:

*If  $ab \leq 2$  for all  $a < 1$ , then  $b \leq 2$ .*

(a) (10 points) State the contrapositive of this statement.

**Solution:**

If  $b > 2$ , then there exists  $a < 1$  such that  $ab > 2$ .

(b) (14 points) Prove the statement. Clearly state what you are assuming and what you want to prove.

**Solution:**

We assume that  $b > 2$ . Then we want to find  $a < 1$  such that  $ab > 2$ . i.e. we want to find  $a < 1$  such that  $a > \frac{2}{b}$ . (Note that  $b > 2$  so  $b \neq 0$ , which means  $\frac{2}{b}$  is a real number.) Thus the question is reduced to proving that if  $b > 2$ , then there exists a number  $a$  between 1 and  $\frac{2}{b}$ .

Well,  $b > 2$ , so  $1 > \frac{2}{b}$ . Thus there are real numbers between 1 and  $\frac{2}{b}$ . Letting  $a$  be any one of those numbers works. (You would receive full credit for explaining this.)

In particular, one such number is the midpoint between  $\frac{2}{b}$  and 1, which is  $\frac{2}{b} + \frac{1 - \frac{2}{b}}{2} = \frac{2+b}{2b}$ . Let  $a = \frac{2+b}{2b}$ .

We can show algebraically that  $a$  is between 1 and  $\frac{2}{b}$ , as follows. Since  $b > 2$ , we have  $b + b > b + 2$ , or  $2b > b + 2$ , which means  $1 > \frac{b+2}{2b}$ . (So  $a < 1$ .) Furthermore,  $b > 2$  means  $\frac{1}{2} > \frac{1}{b}$ . But

$$a = \frac{2+b}{2b} = \frac{1}{b} + \frac{1}{2} > \frac{1}{b} + \frac{1}{b} = \frac{2}{b},$$

so  $a > \frac{2}{b}$ , as desired.

3. (a) (8 points) State the triangle inequality.

If  $c, d \in \mathbb{R}$ , then  $|c + d| \leq |c| + |d|$ .

- (b) (10 points) Prove that  $|b - a| \geq |b| - |a|$ .

**Solution:**

By the triangle inequality, with  $c = b - a$  and  $d = a$ , we get  $|b - a| + |a| \geq |b - a + a|$ . But the RHS is just  $|b|$ . Subtracting  $|a|$  from both sides, we get  $|b - a| \geq |b| - |a|$ , as desired.

- (c) (14 points) Let  $\varepsilon > 0$ , and let  $a, b \in \mathbb{R}$ . Prove if  $|a| < \varepsilon$ , then  $|b| + \varepsilon > |b - a| > |b| - \varepsilon$ .

**Solution:**

Think of this as two questions. First prove that  $|b - a| > |b| - \varepsilon$ , and then prove that  $|b| + \varepsilon > |b - a|$ .

**I. Proving  $|b - a| > |b| - \varepsilon$ :**

Since  $|a| < \varepsilon$ , we have  $-|a| < -\varepsilon$ . So

$$|b| - |a| > |b| - \varepsilon.$$

Thus using part (b) above, we have

$$|b - a| \geq |b| - |a| > |b| - \varepsilon.$$

**II. Proving  $|b| + \varepsilon > |b - a|$ :**

On the other hand,

$$|b| + |a| \geq |b - a|$$

using the triangle inequality with  $c = b$  and  $d = -a$ . So since  $|a| < \varepsilon$ , we get

$$|b| + \varepsilon > |b| + |a| > |b - a|.$$

**III. Conclusion:**

Combining these we get

$$|b| + \varepsilon > |b - a| > |b| - \varepsilon,$$

as desired.

- (d) (2 points) Restate (c) in terms of neighborhoods.

**Solution:**

If  $a \in V_\varepsilon(0)$ , then  $|b - a| \in V_\varepsilon(|b|)$ . Alternatively, in words, if  $a$  is in an  $\varepsilon$ -neighborhood of 0, then  $|b - a|$  is in an  $\varepsilon$ -neighborhood of  $|b|$ .