

1.
 - a. (8 pts) Define a homomorphism of groups.
 - b. (8 pts) Define a homomorphism of rings.
 - c. (10 pts) Let $f : \mathbb{Z}_6 \rightarrow \mathbb{Z}_{24}$ be given by $f([x]_6) = [8x]_{24}$. Prove f is a homomorphism of groups (under $+$) but not a homomorphism of rings.
 - d. (10 pts) Let $f : \mathbb{Z}_8 \rightarrow \mathbb{Z}_{24}$ be given by $f([x]_8) = [9x]_{24}$. Prove f is a homomorphism of rings.
2.
 - (a) (8 pts) Define the kernel of a homomorphism of groups.
 - (b) (16 pts) Prove Proposition 3.3.4, which says: Let G_1 and G_2 be groups, and let $\phi : G_1 \rightarrow G_2$ be a homomorphism of groups. Then ϕ is one-to-one if and only if $\ker(\phi) = \{e\}$.
 - (c) (8 pts) Let $\phi : S_3 \rightarrow \mathbb{Z}_6$ be a homomorphism of groups. Prove that $\ker(\phi) \neq \{(1)\}$, where (1) is the identity in S_3 .
3. Let

$$R = \left\{ \frac{n}{2^m} : n, m \in \mathbb{Z} \right\}.$$

- (a) (10 pts) Show that $(R, +, \cdot)$ is a commutative ring with 1.
- (b) (6 pts) Is R a field? Explain.
- (c) (6 pts) Is R an integral domain? Explain.
4. A subset $I \subseteq R$ of a ring R is called *closed under scalar multiplication* if for all $r \in R$ and all $i \in I$, we have $ri \in I$.
 - (a) (5 pts) Let $I = 2\mathbb{Z}$ and $R = \mathbb{Z}$. Prove I is closed under scalar multiplication.
 - (b) (5 pts) Let R and S be rings and $\phi : R \rightarrow S$ be a homomorphism of rings. Prove $\ker(\phi)$ is closed under scalar multiplication.

Final Exam Cumulative Part

1. Let $G = \{(x, y) : x, y \in \mathbb{R} \text{ with } x \neq 0\}$, and consider the binary operation $*$ given by

$$(x, y) * (z, w) = (xz, xw + y)$$

for $(x, y), (z, w) \in G$.

- (a) (10 pts) Prove that $(G, *)$ is a group.
- (b) (10 pts) Prove that $H = \{(1, y) : y \in \mathbb{R}\}$ is a subgroup of G (remember to use the operation $*$).
2.
 - (a) (8 pts) Prove $\mathbb{Z}_3 \times \mathbb{Z}_2$ is cyclic.
 - (b) (8 pts) Prove $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic.

- (c) (8 pts) Let H be the subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by $([1]_m, [1]_n)$, i.e. $H = \langle ([1]_m, [1]_n) \rangle$. How many elements are in H ? How many different cosets will H have in $\mathbb{Z}_m \times \mathbb{Z}_n$.

3. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 5 & 6 & 8 & 1 & 4 \end{pmatrix}$$

- (a) (5 pts) Write σ in cycle notation as disjoint cycles.
(b) (5 pts) What is the order of σ ?
(c) (5 pts) Is σ odd or even?
(d) (5 pts) Let $H = \{(1), (12)\}$ be a subgroup of S_8 . What is the left coset of H determined by σ ?
4. (16 pts) Let

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}.$$

Prove that the group, A (under matrix multiplication) is isomorphic the group $(\mathbb{R} \setminus \{0\}, \cdot)$ of non-zero real numbers (under multiplication). *You do not need to prove A is a group!*

5. (a) (5 pts) Give an example of an integral domain.
(b) (5 pts) Give an example of a commutative ring that is not an integral domain. Name a zero-divisor in your example.
(c) (5 pts) Give an example of a field.
(d) (5 pts) Give an example of an integral domain that is not a field. Name a non-invertible element in your example.