ON THE PATTERSON-SULLIVAN MEASURE FOR GEOMETRICALLY FINITE GROUPS ACTING ON COMPLEX OR QUATERNIONIC HYPERBOLIC SPACE

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Abstract. The goal of this paper is to provide a tool, the Global Measure Formula, that will facilitate the study of the limit set of discrete geometrically finite groups of isometries of the rank one symmetric spaces. We consider the shadow of a ball from a fixed reference point onto the boundary, and prove a formula that describes the measure of the shadow in terms of the the center of the shadowed ball, generalizing a result from real hyperbolic geometry.

The goal of this paper is to provide a tool, the Global Measure Formula, that will facilitate the study of the limit set of discrete geometrically finite groups of isometries of the rank one symmetric spaces. We consider the shadows of balls in a hyperbolic space on the space's boundary at infinity from a fixed reference point (see Figure 1). The Global Measure Formula relates the Patterson-Sullivan measure of a shadow to the point at which the shadowed ball is centered, providing a tight relationship between the metric and the measure at hand. We give the rigorous statement of the formula in Section 1.

The measure we study here is the Patterson-Sullivan measure on the boundary of complex or quaternionic hyperbolic spaces. Its origins lie with Patterson [21] and Sullivan [30],[31] for real hyperbolic spaces (see also [19] for an introduction), and their work has been continued in this and many other settings, including for example, Gromov hyperbolic spaces [8],[7], general rank 1 manifolds [18] and higher rank symmetric spaces [1]. For other results about this measure in the real rank one symmetric spaces that we study here, see [17], [9],[10],[33].

For real hyperbolic spaces, the Global Measure Formula is attributed to Sullivan [31], and it is proved in the form we use here by B. Stratmann and S. Velani [29], using technical calculations in the upper-half space model of real hyperbolic space. In this paper we generalize the work of Stratmann and Velani to complex and quaternionic hyperbolic spaces. Many of the main ideas follow the original paper, however we have organized the argument more carefully, and in dealing with the difficulties that arise from our more general setting, we have attempted to illuminate some of the more technical calculations. One of the goals of this paper is to demonstrate clearly the central idea in [29] (and [31]), which is to take advantage of the virtually nilpotent structure of the stabilizer of a parabolic fixed point p in a geometrically finite group (see Section 2.5). The result depends on the fact that in a geometrically finite group, this stabilizer acts co-compactly on the limit set minus p as well as on a lie subgroup of the nilpotent stabilizer of p in the full group of

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isometries of hyperbolic space. As usual in generalizing from results in real hyperbolic space to the complex and quaternion hyperbolic spaces, difficulties arise due to not having a conformal class of Riemannian metrics on the boundary; instead, we have a conformal class of sub-Riemannian metrics. We address this partly by using horospheric coordinates instead of the real upper-half space model (see Section 2.2), and by making careful estimates that control the relationship between our shadows and the sub-Riemannian metrics on the boundary (See Section 3.1).

We conclude the introduction by commenting that B. Stratmann has proved many results relying on the real hyperbolic Global Measure Formula as a key lemma. He has used it to study the fine structure of the limit set [23],[24] as well as properties of the geodesics and geodesic flows [25],[26],[27]. It seems likely that many of these results will generalize to the real rank 1 symmetric spaces. For example the following application of the Global Measure Formula due to B. Stratmann and M. Urbanski in the case of real hyperbolic spaces generalizes without significant modification.

Patterson-Sullivan theory has been used to relate the critical exponent of the group to the Hausdorff dimension of the limit set in many settings, which we do not attempt to list here. In [28], B. Stratmann and M. Urbanski calculate the box dimension of the limit set of a geometrically finite group acting on real hyperbolic space, using the Global Measure Formula. The proof generalizes without significant modification, using our generalized version of the Global Measure Formula and Proposition 3.1, which allows us to use covers by shadows to replace the covers by balls in the definition of box dimension; we state it here in our general setting without proof. Here g_O is the sub-Riemannian metric on the boundary of hyperbolic space associated to O.

Theorem. *If Γ is a non-elementary geometrically finite group of isometries acting on real, complex or quaternionic hyperbolic space, then with respect to g_O, the box dimension of the limit set is equal to the critical exponent of the group.*

After this paper was written, I received a personal communication from B. Schapira that a paper generalizing the Global Measure Formula to Hadamard manifolds is in preparation [22].

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1. Statement of the Global Measure Formula

In this section, we list the vocabulary necessary to state the Global Measure Formula, referencing the sections where the technical definitions are given. Let $\mathbb{F}$ be the quaternion, the complex or the real numbers, and let $\mathcal{H}^n_{\mathbb{F}}$ denote $\mathbb{F}$-hyperbolic space, (see Section 2.1 for the definition). Let ρ denote the distance induced by the Riemannian metric and denote the isometry group by $O_{\mathbb{F}}(1,n)$.

Let $O \in \mathcal{H}^n_{\mathbb{F}}$ be a fixed reference point, and let Γ be a discrete subgroup of $O_{\mathbb{F}}(1,n)$. The limit set Λ_Γ is defined to be the set of accumulation points of the orbit ΓO. The limit set does not depend on the choice of $O \in \mathcal{H}^n_{\mathbb{F}}$ ([12]), and because Γ is discrete, Λ_Γ necessarily lies in $\partial\mathcal{H}^n_{\mathbb{F}}$. The group Γ acts properly discontinuously on $\mathcal{H}^n_{\mathbb{F}}$, but not necessarily on $\partial\mathcal{H}^n_{\mathbb{F}}$; the limit set of Γ is the complement of the set of points in $\partial\mathcal{H}^n_{\mathbb{F}}$ on which Γ does act discontinuously. Let $\Omega_\Gamma = \partial\mathcal{H}^n_{\mathbb{F}} \backslash \Lambda_\Gamma$. Then Γ acts discontinuously on $\mathcal{H}^n_{\mathbb{F}} \cup \Omega_\Gamma$, and we form the quotient manifold $M_\Gamma = \mathcal{H}^n_{\mathbb{F}} \cup \Omega_\Gamma/\Gamma$.

Definition 1.1. Let Γ be a discrete subgroup of $O_{\mathbb{F}}(1,n)$, and let $Hull(\Lambda_\Gamma)$ be the convex hull in $\mathcal{H}^n_{\mathbb{F}}$ of the limit set Λ_Γ. Then Γ is convex co-compact if $Hull(\Lambda_\Gamma)/\Gamma$ is compact, and Γ is geometrically finite if it is finitely generated and if, for every $\epsilon > 0$, the volume of an ϵ-neighborhood of $Hull(\Lambda_\Gamma)/\Gamma$ is finite.

For several equivalent definitions of geometrically finite see Bowditch [5]. The resulting Riemannian manifolds are not necessarily of finite volume, but loosely, all the interesting dynamics goes on in a finite volume region. In $\mathcal{H}^2_{\mathbb{R}}$, a discrete group of isometries is geometrically finite if and only if it has a finite sided fundamental domain, however in [15], it is shown that this definition does not extend to $\mathcal{H}^n_{\mathbb{C}}$.

Let δ be the critical exponent of Γ and let μ_O denote the Patterson-Sullivan measure associated to Γ. Then μ_O is a δ conformal density supported on Λ_Γ. In Section 2.6, we give the definition of the critical exponent and the Patterson-Sullivan measure, and indicate the properties we need.

The following geometric structure of M_Γ at infinity is discussed in Section 2.4. Margulis' Lemma (Theorem 2.4) implies that M_Γ can be decomposed into a thick part, where the injectivity radius is greater than a given ε, and a thin part made up of a finite collection of disjoint cusps, where the injectivity radius is less than ε. The constant ε can be chosen such that the lift of the thin part is a collection of disjoint horoballs centered at the parabolic fixed points of Γ, and by choosing ε small, we can make the thin part of M_Γ as small as well like. We call our choice of such a collection the standard horoballs, and for a parabolic fixed point $p \in \partial\mathcal{H}^n_{\mathbb{F}}$, we denote the standard horoball centered at p by $\mathcal{H}_p$. For our theorem, we will need to choose the standard horoballs so that the thin part of M_Γ is small enough; we make that assumption precise in Section 3.2.

In Section 2.5, we define the rank of a parabolic cusp, and briefly describe the corresponding geometry. Since each cusp corresponds to a class of parabolic fixed points in $\partial\mathcal{H}^n_{\mathbb{F}}$, we associate to each parabolic fixed point the rank of its cusp. In $\mathcal{H}^n_{\mathbb{R}}$, this rank is an integer $0 < k \leq n-1$, and in $\mathcal{H}^n_{\mathbb{F}}$ we say the rank is determined by an integral vector (k,l) where $0 < k \leq (n-1)\cdot \dim_{\mathbb{R}} \mathbb{F}$ and $0 \leq l \leq \dim_{\mathbb{R}} \cdot \mathrm{Im}\mathbb{F}$. For a point $z \in \mathcal{H}^n_{\mathbb{F}}$ we define $rank(z)$ to reflect where z is relative to our choice of standard horoballs. If $z \in \mathcal{H}_p$ where p is a parabolic fixed point of Γ with rank (l,k), we let $rank(z) = 2l + k$, and otherwise, if z is in the lift of the thick part of M_Γ, we let $rank(z) = \delta$.

Our work concerns the shadow at infinity of a metric ball in $\mathcal{H}^n_{\mathbb{F}}$. We will use the following notation throughout the paper. Let $O \in \mathcal{H}^n_{\mathbb{F}}$ and $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$, and let ξ_t be the unit speed parameterization of the geodesic connecting O to ξ. We denote by $S_r(O,\xi,R)$ the set of endpoints in $\partial\mathcal{H}^n_{\mathbb{F}}$ of geodesic rays from O that pass through $B(\xi_R, r)$. This can be pictured as the shadow of a solid ball of radius r when the light source O lies a distance R away. We abbreviate the notation to $S(O,\xi,R)$ when $r = 1$. See Figure 1.

The following is Sullivan's Shadow Lemma, proved by Sullivan [30] (see also [19]) in the case of $\mathcal{H}^n_{\mathbb{R}}$, and by Corlette [9] for any rank 1 symmetric space.

Theorem 1.2 (Sullivan's Shadow Lemma). *Let $O \in \mathcal{H}^n_{\mathbb{F}}$ be a fixed reference point. Let Γ be a non-elementary discrete geometrically finite subgroup of $O_{\mathbb{F}}(1,n)$, and let M_Γ be the associated quotient manifold. Let $\bar{M}_{thick}$ denote the lift to $\mathcal{H}^n_{\mathbb{F}}$ of the thick part of M_Γ.*

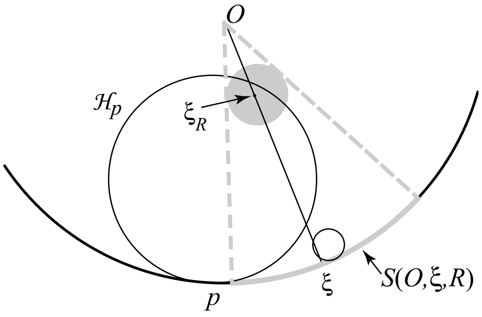

FIGURE 1. The set $S(O, \xi, R)$ is the shadow on $\partial\mathcal{H}^n_{\mathbb{F}}$ of the ball of radius 1 centered at ξ_R.

There exist positive constants c, c' *and* R_o *such that for all* ξ *in* $\Lambda(\Gamma)$, *and* $R > R_o$ *for which* ξ_R *lies in* $\bar{M}_{thick}$, *we have*

$$c \leq \frac{\mu_O(S(O, \xi, R))}{e^{-\delta R}} \leq c'.$$

Our main work is to understand the behavior of the measure of a shadow when ξ_R passes outside of the lift of the thick part of M_Γ, i.e. when ξ_R passes into the cusps. In the case of $\mathcal{H}^n_{\mathbb{R}}$, this was done by Stratmann and Velani [29], using the structure of $\partial\mathcal{H}^n_{\mathbb{R}}$ near the cusps and calculations in the upper half-space model of $\mathcal{H}^n_{\mathbb{R}}$. We generalize the result here to include $\mathcal{H}^n_{\mathbb{F}}$, taking into account the more complicated structure near the cusps and using horospheric coordinates on $\mathcal{H}^n_{\mathbb{F}}$ (see Section 2.1) instead of the upper half-space model of $\mathcal{H}^n_{\mathbb{R}}$. The geometric language of horospheric coordinates will allow us to better understand the geometry behind the technical calculations involved in the argument.

As R increases, the point ξ_R moves along the geodesic from O to ξ, passing in and out of the lifts of the cusps and thick part of M_Γ. We find, as in $\mathcal{H}^n_{\mathbb{R}}$, that as ξ_R passes through the standard horoballs, the measure of the Shadow depends on how far into the standard horoball ξ_R lies (reflected in the $\rho(\xi_R, \Gamma O)$ term in the theorem).

Main Theorem (Global Measure Formula). *Let* $O \in \mathcal{H}^n_{\mathbb{F}}$ *be a fixed reference point. Let* Γ *be a non-elementary discrete geometrically finite subgroup of* $O_{\mathbb{F}}(1, n)$. *Then for a suitable choice of standard horoballs for* Γ, *there exist positive constants* c, c' *such that for all* $\xi \in \Lambda_\Gamma$ *and for all* $R > 1$, *we have*

$$c \leq \frac{\mu_O(S(O, \xi, R))}{e^{-\delta R}\left(e^{\rho(\xi_R, \Gamma O)}\right)^{rank(\xi_R)-\delta}} \leq c'.$$

Notice that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and $R > 1$ such that ξ_R is in $\bar{M}_{thick}$, $rank(\xi_R) - \delta = 0$ and the statement reverts to Sullivan's Shadow Lemma.

We give a brief outline of the rest of the paper. In Section 2, we give the background and notation that we will use throughout. In particular, we define the hyperbolic spaces (Section 2.1), horospheric coordinates as a model for these spaces (Section 2.2), the metrics on their boundaries at infinity (Section 2.3), the structure of the groups and spaces near the parabolic cusps (Sections 2.4 and 2.5) and the Patterson-Sullivan measures on their boundaries at infinity (Section 2.6).

The bulk of our work takes place in Sections 3 and 4. The major differences between the real and the complex and quaternionic hyperbolic geometry begin to show up in the development of the tools in Section 3. In particular, in Section 3.1, we develop relationships between shadows and sub-Riemannian metric balls on the boundary at infinity. In Section 3.2, we choose a set of standard horoballs, centered at parabolic fixed points lying on the boundary of a single fundamental domain. In Section 2.5 we defined what we call a "tile" with which we will estimate the measures of our shadows; in Section 3.3, we make estimates on the measure of translates of this tile and on the number of translates that may lie in sub-Riemannian metric balls. In Section 4, we prove the global measure formula for the standard horoballs defined in Section 3.2, dividing the argument into cases in Section 4.1, and proving the cases Sections 4.2, 4.3 and 4.4.

Finally in Section 5, we use the results Section 4 to prove the Global Measure Formula as stated above.

2. Background and notation

2.1. **The definition of $\mathcal{H}^n_{\mathbb{F}}$.** Let $\mathbb{F}$ be the quaternion, the complex or the real numbers, and let Im$\mathbb{F}$ denote the imaginary part of $\mathbb{F}$: $\mathbb{R}$ if $\mathbb{F}=\mathbb{C}$, and $\mathbb{R}^3$ if $\mathbb{F}$ is the quaternions. Let $\mathbb{F}^{n,1}$ denote a vector space of $\mathbb{F}$-dimension $n+1$, equipped with the Hermitian form $\langle\cdot,\cdot\rangle$ defined by

$$\langle Z,W\rangle = Z_1\overline{W}_1 + Z_2\overline{W}_2 + \cdots + Z_n\overline{W}_n - Z_{n+1}\overline{W}_{n+1}.$$

A non-zero vector $Z\in\mathbb{F}^{n,1}$ is called negative if $\langle Z,Z\rangle<0$, positive if $\langle Z,Z\rangle>0$ and null if $\langle Z,Z\rangle=0$. Let $0\neq\lambda\in\mathbb{F}$, and note that if Z is negative (respectively positive or null) then λZ is also. Thus the definitions of negative, positive and null can be extended to $\mathbb{F}$-lines. Let $[Z]$ denote the line $\{\lambda Z:\lambda\in\mathbb{F}\}$. Then we say $[Z]$ is negative (respectively positive or null) if $\langle Z,Z\rangle<0$ (respectively >0 or $=0$).

The hyperbolic space $\mathcal{H}^n_{\mathbb{F}}$ is defined to be the collection of negative $\mathbb{F}$-lines in $\mathbb{F}^{n,1}$, and its boundary $\partial\mathcal{H}^n_{\mathbb{F}}$, the collection of null $\mathbb{F}$-lines. The tangent space of $\mathcal{H}^n_{\mathbb{F}}$ to $[Z]$ can be identified to the subspace in $\mathbb{F}^{n+1}$ that is orthogonal to Z (with respect to $\langle\cdot,\cdot\rangle$). The restriction of $\langle\cdot,\cdot\rangle$ to this subspace is positive definite, and hence defines a Riemannian metric, called the Bergman metric. We normalize the metric to have sectional curvature pinched between $-\frac{1}{4}$ and -1, and we have the following formula for the distance $\rho([Z],[W])$ between $[Z],[W]\in\mathcal{H}^n_{\mathbb{F}}$:

$$\cosh^2\left(\frac{\rho([Z],[W])}{2}\right) = \frac{\langle Z,W\rangle\langle W,Z\rangle}{\langle Z,Z\rangle\langle W,W\rangle}, \tag{1}$$

which is independent of the choice of vectors spanning the $\mathbb{F}$-lines $[Z]$ and $[W]$. For more about complex hyperbolic space, we refer the reader to [14].

In [10], the authors prove a useful lemma about the behavior of the distance ρ.

Lemma 2.1. [10]*[Lemma 3.1] For any $\epsilon>0$, there is some constant C such that for any geodesic triangle in $\mathcal{H}^n_{\mathbb{F}}$ with sides a,b,c and opposite angles α,β,γ with $\gamma>\epsilon$, we have $a+b-C\le c\le a+b$.*

Let $O\in\mathcal{H}^n_{\mathbb{F}}$ be a fixed reference point. The isometry group of $\mathcal{H}^n_{\mathbb{F}}$ is $O_{\mathbb{F}}(1,n)$, the subgroup of $GL(n+1,\mathbb{F})$ that preserves the signature $(1,n)$ inner product on $\mathbb{F}^{n+1}$. The stabilizer in $O_{\mathbb{F}}(1,n)$ of $O\in\mathcal{H}^n_{\mathbb{F}}$ is a maximal compact subgroup K, and $\mathcal{H}^n_{\mathbb{F}}$ can be identified as a homogeneous space with $O_{\mathbb{F}}(1,n)/K$. Taking an Iwasawa decomposition (see for example [11] 2.19.3) of $O_{\mathbb{F}}(1,n)$, we have $O_{\mathbb{F}}(1,n)=KAN$, where A and N fix a point $p \in \partial\mathcal{H}^n_{\mathbb{F}}$. Then $\mathcal{H}^n_{\mathbb{F}}$ is the principal homogeneous space for the semi-direct product AN. As groups, A is isomorphic to the multiplicative group of real numbers, and N is a nilpotent group. The projection $AN \to A$ induces a function $\mathcal{H}^n_{\mathbb{F}} \to \mathbb{R}$ that is well defined up to overall multiplication by elements of A. This class of functions corresponds to the class of Busemann functions at p, given by

$$b^O_p(x) = \lim_{t\to\infty} \rho(p_t, x) - t,$$

where p_t is the unit speed geodesic connecting O to p (where overall multiplication by elements of A corresponds to changing O to another point on the geodesic connecting O to p). The horospheres centered at p are the orbits of N in $\mathcal{H}^n_{\mathbb{F}}$, which coincide with level sets of the Busemann function.

2.2. **Heisenberg and horospheric coordinates.** Denote the horosphere centered at p and through O by $H_p(O)$. Then $H_p(O)$ is a submanifold of real dimension $n\cdot dim_{\mathbb{R}}\mathbb{F}-1$, and hence its tangent space at O, $T_OH_p(O)$, sits as a hyperplane in $T_O\mathcal{H}^n_{\mathbb{F}} \cong \mathbb{F}^n$. We describe the construction of distributions on the horospheres centered at p induced by the $\mathbb{F}$ structure on $\mathcal{H}^n_{\mathbb{F}}$ (See [20] for a complete account). Let $w \in (T_OH_p(O))^\perp$ be the outward normal unit vector to $H_p(O)$ at O. Since $\mathbb{F}$ acts on the vector space $T_O\mathcal{H}^n_{\mathbb{F}}$ by scalar multiplication, $(\mathbb{F}w)$ is a subspace of $T_O\mathcal{H}^n_{\mathbb{F}}$ of real dimension $dim_{\mathbb{R}}\mathbb{F}$. We have

$$T_OH_p(O) = (\mathbb{F}w)^\perp \oplus \mathbb{F}w \cap T_OH_p(O),$$

with $(\mathbb{F}w)^\perp \cong \mathbb{F}^{n-1}$ and $\mathbb{F}w \cap T_OH_p(O) \cong Im(\mathbb{F})$.

For each $y \in \mathcal{H}^n_{\mathbb{F}}$ decompose $T_yH_p(y)$ into subspaces V^1_y and V^2_y, where $V^1_y = (\mathbb{F}(v_y))^\perp$, and $V^2_y = \mathbb{F}v_y \cap T_yH_\xi(y)$. The resulting distributions, V^1 and V^2, are invariant under the actions of both N and A, thus they can be extended by projecting along geodesics asymptotic to p to yield distributions on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ invariant under the action of A and N.

In fact, the distribution V^1 and its orthogonal complement V^2 on $\partial\mathcal{H}^n_{\mathbb{F}}$ (extended continuously to p) are independent of the choice of $p \in \partial\mathcal{H}^n_{\mathbb{F}}$ and invariant under the action of the full group $O_{\mathbb{F}}(1,n)$ of isometries of $\mathcal{H}^n_{\mathbb{F}}$ [20]. Moreover, V^1 and V^2 are the only nontrivial distributions on $\partial\mathcal{H}^n_{\mathbb{F}}$ invariant under the action of $O_{\mathbb{F}}(1,n)$ [33]; in particular, in the case that $\mathbb{F} = \mathbb{R}$, there are no non-trivial $O_{\mathbb{R}}(1,n)$-invariant distributions on $\partial\mathcal{H}^n_{\mathbb{R}}$ [13].

Following Goldman and Parker [15], we use these distributions to construct Heisenberg coordinates on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$, and then horospheric coordinates on $\mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ that mimic the upper half plane model in real hyperbolic geometry. Let $p \in \partial\mathcal{H}^n_{\mathbb{F}}$ be fixed and let $O_\partial \in \partial\mathcal{H}^n_{\mathbb{F}}$ be the endpoint of the geodesic from p through O.

We have $T_{O_\partial}\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\} \cong \mathbb{F}^{n-1} \oplus \mathrm{Im}\mathbb{F}$. The exponential map at O_∂ identifies $T_{O_\partial}\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ to $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$, allowing us to assign coordinates in $\mathbb{F}^{n-1} \oplus \mathrm{Im}\mathbb{F}$ on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$, with origin $(0,0) = O_\partial$. We also have $N \cong \mathbb{F}^{n-1} \oplus \mathrm{Im}\mathbb{F}$ by identifying N to its orbit $NO_\partial = \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$. These coordinates on N and $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ are called Heisenberg coordinates, since in these coordinates, N has the structure of the Heisenberg group: for $(\zeta_1, v_1), (\zeta_2, v_2) \in N \cong \mathbb{F}^{n-1} \oplus \mathrm{Im}\mathbb{F}$, we have

$$(\zeta_1, v_1)(\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\mathrm{Im} \ll \zeta_1, \zeta_2 \gg), \tag{2}$$

where $\ll \cdot, \cdot \gg$ is the positive definite Hermitian form on $\mathbb{F}^{n-1}$.

The horospheric coordinates on $\mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$ are defined from the Heisenberg coordinates. As in Goldman and Parker, we describe explicitly how to get from a line through a nonpositive vector $Z \in \mathbb{F}^{n,1}$ to horospheric coordinates $(\zeta, v, u) \in \mathbb{F}^{n-1} \oplus \mathrm{Im}\mathbb{F} \oplus \mathbb{R}^+$, for a convenient choice of origin $O \in \mathcal{H}^n_{\mathbb{F}}$ and point at infinity $p \in \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$. Let $0' = (0, \ldots, 0)$ be the zero vector in $\mathbb{F}^{n-1}$, let

$$O = \left[\begin{pmatrix} 0' \\ 0 \\ 1 \end{pmatrix}\right]$$

be the origin in $\mathcal{H}^n_{\mathbb{F}}$, and let

$$p = \left[\begin{pmatrix} 0' \\ -1 \\ 1 \end{pmatrix}\right]$$

be the point at infinity in $\partial\mathcal{H}^n_{\mathbb{F}}$. Let $x \in \mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}}$. Then coordinates $(\zeta, v, u) \in \mathbb{F}^{n-1} \times \mathrm{Im}\mathbb{F} \times \mathbb{R}^+$ for x are defined by letting $(\zeta, v) \in \mathbb{F}^{n-1} \times \mathrm{Im}\mathbb{F}$ be the coordinates of the element $n \in N$ with $nO = x$, and letting $u = e^{-b^O_p(x)}$.

Recall that $\mathrm{Im}\mathbb{F} = \mathbb{R}^k$ where $k = \dim_{\mathbb{R}} \mathbb{F} - 1$. Let $I = 0$ when $\mathbb{F} = \mathbb{R}$, $I = i \in \mathbb{C}$ when $\mathbb{F} = \mathbb{C}$ and $I = (i, j, k)$, when $\mathbb{F}$ is the quaternion numbers. Then for $v \in \mathrm{Im}\mathbb{F}$, $I \cdot v$ is an element of $\mathbb{F}$ with zero real part. Explicitly, the identification $\mathcal{H}^n_{\mathbb{F}} \to \mathbb{F}^{n-1} \times \mathrm{Im}\mathbb{F} \times \mathbb{R}^+$ is given by

$$\left[\begin{pmatrix} Z' \\ Z_n \\ Z_{n+1} \end{pmatrix}\right] \mapsto (\zeta, v, u), \tag{3}$$

where

$$\zeta = \frac{Z'}{Z_n + Z_{n+1}}$$

$$v = \mathrm{Im}\left(\frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}}\right)$$

$$u = -Re\left(\frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}}\right) - \frac{\ll Z', Z' \gg}{|Z_n + Z_{n+1}|^2}$$

Its inverse $\mathbb{F}^{n+1} \times \mathrm{Im}\mathbb{F} \times \mathbb{R}^+ \to \mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}}$ is given by

$$(\zeta, v, u) \mapsto \left[\begin{pmatrix} \zeta \\ \frac{1}{2}(1 - \ll \zeta, \zeta \gg - u + I \cdot v) \\ \frac{1}{2}(1 + \ll \zeta, \zeta \gg + u - I \cdot v) \end{pmatrix}\right].$$

We call these coordinates the horospheric coordinates on $\mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$.

These coordinates are particularly convenient when considering projections from p, for moving a point (ζ, v, u) along a geodesic asymptotic to p amounts simply to changing u. Note that for points on $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$, $u = 0$ and for points on $H_p(O)$, $u = 1$. Projection along geodesics asymptotic to p induces a natural identification $\pi_p : H_p(O) \to \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$. We will use the following calculation; its significance is that the distance studied below depends only on one variable ($Z_n \in \mathbb{F}$).

Claim 2.2. *Let* $z = [Z] \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$*, with* $Z = (Z', Z_n, 1) \in \mathbb{F}^{n,1}$ *where* $Z' \in \mathbb{F}^{n-1}$ *and* $Z_n \in \mathbb{F}$*. Then*

$$\pi_p^{-1}(z) = \left[\left(\begin{array}{c} \frac{Z'}{Z_n+1} \\ \frac{1}{2}\left(\frac{Z_n-1}{Z_n+1}\right) \\ \frac{1}{2}\left(2 - \frac{Z_n-1}{Z_n+1}\right) \end{array}\right)\right],$$

and we have

$$\cosh^2 \frac{\rho(O, \pi_p^{-1}(z))}{2} = \frac{1}{4}\left|\frac{2}{Z_n+1} + 1\right|^2.$$

2.3. **Metrics on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$.** In $\mathcal{H}^n_{\mathbb{R}}$, each choice of a point $p \in \partial\mathcal{H}^n_{\mathbb{R}}$ yields an upper half space model in which the induced metrics on horospheres centered at p are Euclidean. These metrics scale uniformly from horosphere to horosphere centered at p, and after scaling converge to a Riemannian metric on $\partial\mathcal{H}^n_{\mathbb{R}}\backslash\{p\}$, and the metrics resulting from different choices of p are conformally related. Though our horospheric coordinates mimic the upper half space model, the induced metric on horospheres are not Euclidean and do not scale uniformly and there is no conformal class of Riemannian metrics on $\partial\mathcal{H}^n_{\mathbb{F}}$. However, we do have uniform scaling in both distributions V^1 and V^2, and the procedure produces a conformal class of subriemannian metrics called the Carnot-Caratheodory metrics. The Heisenberg structure of $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ allows the construction of a distance called the Cygan metric that is equivalent in a metric space sense to the Carnot-Caratheodory metric and both are preserved by the action of the stabilizer in $O_{\mathbb{F}}(1,n)$ of N on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$. The following is a brief description of the construction of both the Carnot-Caratheodory and Cygan metrics. For a clear introduction to these metrics and a theorem giving a concrete coordinate expression for this equivalence in $\mathcal{H}^2_{\mathbb{C}}$, see [4]. As the authors of [4] point out, the Carnot-Caratheodory metric's infinitesimal nature makes it well suited for analytic considerations, while the Cygan metric's algebraic nature makes it appropriate for computations in local coordinates on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$; we will use them each accordingly.

Let $x \in \mathcal{H}^n_{\mathbb{F}}$, let $u \in T_x H_p(x)$ and let $c(t)$ be the geodesic through x and asymptotic to p. Let u_t be the Jacobi field along c with $u_0 = u$. If $u \in V^1_x$, then u_t and $c'(t)$ span a totally real subspace of $T_{c(t)}\mathcal{H}^n_{\mathbb{F}}$ (i.e. u_t and $c'(t)$ span a subspace of $T_{c(t)}\mathcal{H}^n_{\mathbb{F}}$ on which the Hermitian inner product given by the metric is totally real). Thus for all t, u_t lies in the tangent space of a copy of $\mathcal{H}^2_{\mathbb{R}}$, which can be seen to have constant curvature $-\frac{1}{4}$, and the length of u_t is

$$\|u_t\| = e^{\frac{t}{2}}\|u\|.$$

If $u \in V^2_x$, then u_t is tangent to an $\mathbb{F}$-line containing $c'(t)$; since such lines are totally geodesic and of constant curvature -1, we have

$$\|u_t\| = e^t\|u\|.$$

Thus, in the decomposition $TH_p(c(t)) = V^1 \oplus V^2$, the metric m_t induced by the residual Riemannian metric has the form

$$m_t = \left(\begin{array}{cc} e^t & 0 \\ 0 & e^{2t} \end{array}\right).$$

If we normalize by e^{-t} and let $t \to \infty$, the metrics $e^{-t}m_t$ converge to a Riemannian metric when restricted to V^1, but diverge when restricted to V^2. Hence on $\partial\mathcal{H}^n_{\mathbb{F}}$, we have a subriemannian metric, which is Riemannian in the case $\mathbb{F} = \mathbb{R}$ (since $V^2 = \{0\}$) and yields the C-R structure when $\mathbb{F} = \mathbb{C}$ (since $dim_{\mathbb{R}} V^2 = 1$). In [20], Pansu shows that the Riemannian metrics induced on V^1 by different choices of p are conformally related. This produces a conformal class of subriemannian metrics called the Carnot-Caratheodory metrics (see for example [16]).

The Cygan metric is a distance on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ considered as the Heisenberg group. Fix $p \in \partial\mathcal{H}^n_{\mathbb{F}}$ and an origin $O \in \mathcal{H}^n_{\mathbb{F}}$, and let $(\zeta, v) \in \mathbb{F}^{n-1} \oplus \mathrm{Im}\mathbb{F}$ be a point in $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ in Heisenberg coordinates. To define the Cygan metric, we start with the Heisenberg psuedonorm, $|\cdot|_{\mathcal{H}}$ given by

$$|(\zeta, v)|_{\mathcal{H}} = (|\zeta|^4 + |v|^2)^{\frac{1}{4}}, \tag{4}$$

where $|\cdot|$ is the modulus in $\mathbb{F}$ or $\mathbb{F}^{n-1}$, as appropriate. This induces the Cygan metric d_p on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$: let $(\zeta_1, v_1), (\zeta_2, v_2) \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$. Then

$$d_p((\zeta_1, v_1), (\zeta_2, v_2)) = \left| T^{-1}_{(\zeta_1, v_1)}(\zeta_2, v_2) \right|_{\mathcal{H}},$$

where $T_{(\zeta_1, v_1)}$, called the Heisenberg translation by (ζ_1, v_1), is given by

$$T_{(\zeta_1, v_1)}(\zeta_2, v_2) = (\zeta_1, v_1)(\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\mathrm{Im} \ll \zeta_1, \zeta_2 \gg), \tag{5}$$

as in Equation (2).

The stabilizer in $O_{\mathbb{F}}(1, n)$ of p is a maximal parabolic subgroup with Langlands decomposition MAN, where M is the stabilizer of p in K. The action of MAN on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ can be described through the action of its generating elements: M acts by rotations, A by dilations and N by translations. The set of isometries of the Carnot-Caratheodory and Cygan metrics on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ is the stabilizer in $O_{\mathbb{F}}(1, n)$ of N, and consists of rotations and translations, generating the semidirect product MN. Since $A \cong \mathbb{R}^*$, the multiplicative group of positive real numbers, we consider elements of A as real numbers of the form e^λ, where $\lambda \in \mathbb{R}$. The action of $e^\lambda \in A$ on $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ in Heisenberg coordinates is given by

$$e^\lambda(\zeta, v) \mapsto (e^\lambda \zeta, e^{2\lambda} v). \tag{6}$$

The exponents for the Heisenberg pseudonorm were chosen to scale homogeneously with respect to this dilation; we have

$$d_p(e^\lambda(\zeta_1, v_1), e^\lambda(\zeta_2, v_2)) = e^\lambda d_p((\zeta_1, v_1), (\zeta_2, v_2))$$

for $\lambda \in \mathbb{R}$ and $(\zeta_1, v_1), (\zeta_2, v_2) \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$.

2.4. **Parabolic fixed points and standard horoballs.**

We will consider the action of a discrete subgroup Γ of $O_{\mathbb{F}}(1, n)$ on Λ_Γ.

Definition 2.3.

1. An element $\gamma \in O_{\mathbb{F}}(1, n)$ is called parabolic if the set of points in $\mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}}$ fixed by γ consists of a single point $p \in \partial\mathcal{H}^n_{\mathbb{F}}$.
2. A subgroup G of $O_{\mathbb{F}}(1, n)$ is called parabolic if the set of points in $\mathcal{H}^n_{\mathbb{F}} \cup \partial\mathcal{H}^n_{\mathbb{F}}$ fixed by G consists of a single point $p \in \partial\mathcal{H}^n_{\mathbb{F}}$, and G preserves setwise any horosphere based at p.
3. If $p \in \partial\mathcal{H}^n_{\mathbb{F}}$ is fixed by a non-trivial parabolic subgroup of G, then p is called a parabolic fixed point of G.

Margulis' Lemma provides a universal constant $\bar{\varepsilon}$ depending only on the curvature bounds and the dimension that defines a decomposition of M_Γ into its thick and thin parts by cutting off the cusps at horospheres based at parabolic fixed points. See, for example, [5] for a full account.

Theorem 2.4 (Margulis' Lemma)**.** *For any integer n and for $\kappa \geq 1$, there exists a constant $\bar{\varepsilon} = \bar{\varepsilon}(n, \kappa) > 0$ such that if X is an n-dimensional manifold with sectional curvature bounded between $-\kappa^2$ and -1, and Γ is a discrete subgroup of the isometries of X, and $x \in X$, then the group $\Gamma_\varepsilon(x)$ generated by $\{\gamma \in \Gamma : \rho(x, \gamma x) \leq \varepsilon\}$ contains a nilpotent subgroup of finite index, for all $\varepsilon \leq \bar{\varepsilon}$.*

We apply this to M_Γ. For each $\varepsilon < \bar{\varepsilon}$, define the "$\varepsilon$-thin part" $M_\Gamma^{\varepsilon-thin}$ of M_Γ, by

$$M_\Gamma^{\varepsilon-thin} = \{x \in M_\Gamma : \text{injectivity radius at } x < \varepsilon\},$$

and the "ε-thick part" $M_\Gamma^{\varepsilon-thick}$ of M_Γ, by

$$M_\Gamma^{\varepsilon-thick} = \{x \in M_\Gamma : \text{injectivity radius at } x \geq \varepsilon\}.$$

Margulis' Lemma implies that for ε small enough, the lift of $M_\Gamma^{\varepsilon-thin}$ to $\mathcal{H}_\mathbb{F}^n$ consists of disjoint horoballs, centered at parabolic fixed points of Γ. We call a such a collection of disjoint horoballs "standard horoballs" for the action of Γ; for each parabolic fixed point p, let $\mathcal{H}_p$ denote the standard horoball based at p, and Γ_p denote the stabilizer in Γ of p. In Section 3.2, we explain how we choose the ε that determines the standard horoballs in the Global Measure Formula. The following facts follow from Margulis' Lemma.

Proposition 2.5. [5] *(Propositions 3.2.1 and 4.1.) Let $\Gamma \subset O_\mathbb{F}(1, n)$ be discrete, and let $p \in \partial\mathcal{H}_\mathbb{F}^n$ be a parabolic fixed point of Γ. Then Γ_p is parabolic and virtually nilpotent.*

Definition 2.6. A point $\xi \in \Lambda_\Gamma$ is called radial if any geodesic ray asymptotic to ξ lies within a bounded distance of an infinite number of points of the orbit ΓO (and hence any orbit of Γ).

Theorem 2.7. [10]*(Theorem 2.3.) If Γ is geometrically finite, then the limit set Λ_Γ consists of the radial limit set, together with a countable number of parabolic fixed points.*

Proposition 2.8. [5]*(See Section 5.) If Γ is geometrically finite and $p \in \Lambda_\Gamma$ is a parabolic fixed point of Γ, then $(\Lambda_\Gamma \backslash \{p\})/\Gamma_p$ is compact.*

2.5. The rank and structure of a parabolic cusp (defining the tile Q).

Let p be a parabolic fixed point in Λ_Γ. By Proposition 2.5, the stabilizer Γ_p in Γ of p is parabolic, and hence preserves pointwise any horosphere centered at p. Therefore in the Langlands MAN decomposition of the stabilizer in $O_\mathbb{F}(1, n)$ of p, Γ_p is contained in the semidirect product MN. We would like to take advantage of the Heisneberg structure on N defined in Section 2.1 to study Γ_p, but since it lies in MN rather than N, we cannot do so immediately. To rectify this, we call on a theorem of Bieberbach on crystallographic groups (see for example [32] page 100) extended to the nilpotent setting by L. Auslander [3] (see also [2] Theorem 3.5 in the complex case, [10] Lemma 3.4 for a proof in this setting, or the survey of related results [6]). This theorem tells us that though Γ_p does not sit in N, it has a finite index subgroup whose orbits coincide with those of a discrete group that does sit inside N.

Theorem 2.9 (The Bieberbach Theorem.)**.** *Let Γ be a discrete subgroup of $O_\mathbb{F}(1, n)$, let $p \in \partial\mathcal{H}_\mathbb{F}^n$ be a parabolic fixed point of Γ, and let Γ_p be the stabilizer in Γ of p. Then there exists a finite index subgroup $\Gamma_1 \in \Gamma_p$ and a discrete subgroup $\Gamma_2 \in N$ such that*

(a) Γ_1 *is isomorphic to* Γ_2*;*
(b) Γ_2 *is co-compact in the connected component of the identity of its Zariski closure, denoted* $\bar{\Gamma}_2^Z$*;*
(c) Γ_1 *acts freely by translations on* $\bar{\Gamma}_2^Z$*, (in fact the orbits of* Γ_1 *on* $\bar{\Gamma}_2^Z$ *coincide with those of* Γ_2*).*

Let $\Gamma_1 \subseteq \Gamma_p$ and $\Gamma_2 \subseteq N$ be given by the Theorem, and let $H = \bar{\Gamma}_2^Z$ be the identity component of the Zariski closure of Γ_2. Then H is a nilpotent Lie subgroup of N. Let $Z(H)$ denote the center of H. Let $l = dim_{\mathbb{R}}(Z(H) \cap \mathrm{Im}\mathbb{F})$, and let $k = dim_{\mathbb{R}} H/(Z(H) \cap \mathrm{Im}\mathbb{F})$. In the case that $\mathbb{F} = \mathbb{R}$, $\mathfrak{n} \cong \mathbb{R}^{n-1}$, l is zero and k is called the rank of the parabolic cusp at p. In general, the notion of rank is characterized by the ordered pair (l, k), defined above. Note that l is at most $dim_{\mathbb{R}}\mathrm{Im}\mathbb{F}$, and k is at most $(n-1) \cdot dim_{\mathbb{R}}\mathbb{F}$.

As in the proof of the shadow lemma in the case of $\mathcal{H}^n_{\mathbb{R}}$ [29], the Bieberbach theorem allows us to define a tile Q that behaves like the fundamental domain for the action of Γ_p for the part of $\partial\mathcal{H}^n_{\mathbb{F}}$ in which Λ_Γ (the support of μ_O) lies. To study the μ_O measure of a set $A \subseteq \partial\mathcal{H}^n_{\mathbb{F}}$, we will count how many tiles gQ (for $g \in \Gamma_p$) lie in A and multiply it by an estimate for the measure of such tiles.

Let Γ be a discrete group of isometries of $\mathcal{H}^n_{\mathbb{F}}$ and let p be a parabolic fixed point for Γ, with stabilizer Γ_p. Let Γ_1, Γ_2, H be given by Theorem 2.9, and let h and k determine the rank of p. The Bieberbach Theorem says that H/Γ_p is compact. Since $\Lambda_\Gamma \backslash \{p\}/\Gamma_p$ is also compact, we can find a connected, relatively compact, open subset $Q \subseteq \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$ of smallest possible diameter q (with respect to the cygan metric on $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$) such that the orbit $\Gamma_p Q$ covers both Λ_Γ and H. Furthermore, we can choose Q such that $Q \cap H$ is a fundamental domain for the action of Γ_1 on H. Without loss of generality, we assume the origin O_∂ (corresponding to the end point in $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$ $\sigma_{p,O}(+\infty)$ of the geodesic connecting p to O) lies in Q.

2.6. The Patterson-Sullivan measures on $\partial\mathcal{H}^n_{\mathbb{F}}$.

In this section, we give the definition of the Patterson-Sullivan measure. We refer to [9] and [33], as they discuss the case of $\mathcal{H}^n_{\mathbb{F}}$.

Let $O \in \mathcal{H}^n_{\mathbb{F}}$ be fixed. Given a discrete subgroup Γ of $O_{\mathbb{F}}(1, n)$, the critical exponent $\delta = \delta_\Gamma$ is defined to be the infimum of all s for which the Poincaré series

$$\sum_{\gamma \in \Gamma} e^{-s\rho(x, \gamma y)}$$

converges for some $x, y \in \mathcal{H}^n_{\mathbb{F}}$. This value can be shown to depend only on the group Γ [12].

Relating the critical exponent to the structure we defined in Section 2.5 we have the following lemma.

Lemma 2.10. [10]*(Lemma 3.5) Let* Γ *be geometrically finite with critical exponent* δ*, and let* (k, l) *be the rank of a parabolic fixed point of* Γ*. Then* $2\delta > 2l + k$*.*

Now consider the family of measures

$$\mu_{O,s} = \frac{\sum_{\gamma \in \Gamma} e^{-s\rho(O, \gamma y)} \Delta_{\gamma y}}{\sum_{\gamma \in \Gamma} e^{-s\rho(O, \gamma y)}},$$

where Δ_p is the Dirac measure at a point $p \in \mathcal{H}^n_{\mathbb{F}}$, and y is some fixed reference point. Then $\mu_{O,s}$ is supported on Γy. After reweighing the coefficients using the methods of Patterson [21](see also [19]), we can be assured that the denominator diverges at δ, and there is a subsequence in s which will converge weak* as $s \to \delta$ to a finite measure μ_O supported on the limit set, Λ_Γ.

Proposition 2.11. [9] *(Proposition 5.1) For $x, y \in \mathcal{H}^n_{\mathbb{F}}$, the Patterson-Sullivan measures μ_x and μ_y are mutually absolutely continuous and the following hold.*

(i) The Radon-Nikodym derivative is given by

$$\frac{d\mu_x}{d\mu_y}(\eta) = e^{-\delta b^y_\eta(x)}.$$

(ii) For any $\gamma \in \Gamma$, we have $\gamma^* \mu_x = \mu_{\gamma x}$.

Theorem 2.12. [10]*(Theorem 3.6) The Patterson-Sullivan measures have no atoms.*

3. The main tools

3.1. **Balls vs. Shadows.** The Global Measure Formula concerns the shadow $S(O, \xi, R)$ of a ball of radius 1 centered at the point a distance R from O on the geodesic connecting O to ξ. Recall that when we need to discuss the shadow of a ball of radius r other than 1, we denote it $S_r(O, \xi, R)$. To make our computations, it will be useful to estimate the size of the shadows in terms of the metric on $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$. In fact we will also do the reverse: we will need our knowledge of the metric in $\mathcal{H}^n_{\mathbb{F}}$ involved in constructing the shadow to understand the metric structure on the boundary. The two Propositions proved here allow us to compare our shadows to metric balls on $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$. They are in the spirit of the following proposition by Corlette ([9], Section 2), indicating that computing Hausdorff dimension on $\partial\mathcal{H}^n_{\mathbb{F}}$ using Carnot-Caratheodory balls is equivalent to using shadows. Corlette notes that a similar result is obtained by Pansu [20].

Proposition 3.1 ([9] (Theorem 2.2))**.** *Fix $O \in \mathcal{H}^n_{\mathbb{F}}$, $r > 0$ and a Carnot-Caratheodory or Cygan metric on $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$. There are positive constants A_1 and A_2 such that, for R large enough,*

$$B(\xi, A_1 e^{-R}) \subset S_r(O, \xi, R) \subset B(\xi, A_2 e^{-R}),$$

where $B(\xi, t)$ denotes a ball with respect to the Carnot-Caratheodory or Cygan metric about $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and of radius t.

In the following, we use Carnot-Carathedory or Cygan metric balls to approximate shadows of balls centered at points inside a horosphere based at p, but such that p itself does not lie in the shadow. (This is the case in which the Cygan or Carnot-Carathedory balls of finite radius approximate our shadows.)

Proposition 3.2. *Fix $O \in \mathcal{H}^n_{\mathbb{F}}$, $p \in \partial\mathcal{H}^n_{\mathbb{F}}$, $r > 0$ and a Carnot-Caratheodory or Cygan metric on $\partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$. Let $\epsilon < r$, and let $\mathcal{N}_{r+\epsilon}$ be an $r + \epsilon$ neighborhood of the geodesic from p to O.*

Then there exists constants A_1 and A_2 such that

$$B(\xi, A_1 e^d) \subseteq S_r(O, \xi, R) \subseteq B(\xi, A_2 e^d),$$

for any $\xi \in \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$ and $R > r$ such that ξ_R lies inside the horoball centered at p and through O, but outside $\mathcal{N}_{r+\epsilon}$ (i.e. $\xi_R \in H_p(O) \cap \mathcal{N}^c_{r+\epsilon}$), and where d is the horospheric distance from ξ_R to O (i.e. $d = -b^O_p(\xi_R)$). Here, $B(\xi, t)$ denotes the Carnot-Caratheodory or Cygan metric ball of radius t about ξ.

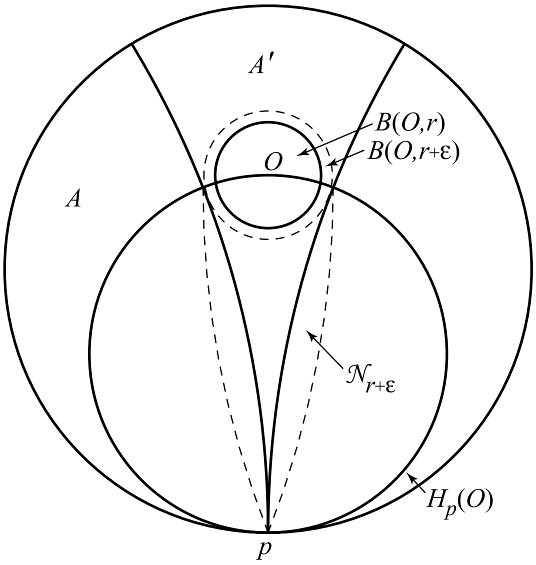

FIGURE 2

Proof. Let A' be union of the geodesics asymptotic to p and through the ball of radius $r+\epsilon$ centered at O, and let A be the points of $\mathcal{H}^n_{\mathbb{F}}$ which are outside A' and outside the horoball centered at p and through O. Then the shadows from points of A of the ball of radius r about O all lie in a compact subset of $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$. See Figure 2.

Let $x \in A$, and let $\xi_x \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ be the endpoint of the geodesic from x through O. We consider the shadows $S_r(x,\xi_x,\rho(x,O))$ from x of the ball of radius r about O. Let $A_1(x)$ be the radius of the largest ball in $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ about ξ_x contained in the shadow $S_r(x,\xi_x,\rho(x,O))$, and let $A_2(x)$ be the radius of the smallest ball in $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ about ξ_x containing the shadow $S_r(x,\xi_x,\rho(x,O))$, where the balls are taken with respect to the Carnot-Caratheodory metric on $\partial\mathcal{H}^n_{\mathbb{F}}$. For $x \in A$, both $A_1(x)$ and $A_2(x)$ are bounded above and below by positive constants. Let $A_1, A_2 \in \mathbb{R}$ such that $0 < A_1 < A_1(x) < A_2(x) < A_2$ for all $x \in A$. We have

$$B(\xi_x, A_1) \subseteq S_r(x,\xi_x,\rho(x,O)) \subseteq B(\xi_x, A_2),$$

for all $x \in A$.

Let $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and $R > r$ such that ξ_R lies inside the horoball centered at p and through O, but $p \notin S_r(O,\xi,R)$, and let $d = -b_p^O(\xi_R)$ be the horospheric distance from ξ_R to O. Let $\sigma_{\xi,R}$ be a loxodromic isometry translating along the geodesic asymptotic to p and through O a distance d away from p. Then $\sigma_{\xi,R}(\xi_R)$ lies on $H_p(O)$ and remains outside $\mathcal{N}_{r+\epsilon}$, since $\partial N_{r+\epsilon}$ is preserved by $\sigma_{\xi,R}$. Let $n_{\xi,R}$ be the element of N that takes $\sigma_{\xi,R}(\xi_R)$ to O. Then $n_{\xi,R}\circ\sigma_{\xi,R}$ takes $S_r(O,\xi,R)$ to $S(x,\xi_x,\rho(x,O))$, where $x = n_{\xi,R}\circ\sigma_{\xi,R}(O) \in A$. So we have

$$B(n_{\xi,R}\circ\sigma_{\xi,R}(\xi), A_1) \subseteq n_{\xi,R}\circ\sigma_{\xi,R}S_r(O,\xi,R) \subseteq B(n_{\xi,R}\circ\sigma_{\xi,R}(\xi), A_2).$$

Hence

$$B(\xi, A_1e^d) \subseteq S_r(O,\xi,R) \subseteq B(\xi, A_2e^d),$$

since N acts on the Carnot-Caratheodory metric by isometries, and $\sigma_{\xi,R}^{-1}$ acts by dilation by a factor of e^d. ☐

Proposition 3.3. *Fix $O \in \mathcal{H}_{\mathbb{F}}^n$, $p \in P$, $r > 0$ and a Carnot-Caratheodory or Cygan metric on $\partial\mathcal{H}_{\mathbb{F}}^n \backslash \{p\}$. Then there exists constants A_1 and A_2 such that for $R > r$,*

$$B(O_\partial, A_1 e^R) \subseteq S_r(O, p, R)^c \subseteq B(O_\partial, A_2 e^R),$$

where $B(\xi, t)$ denotes the Carnot-Caratheodory subriemannian or Cygan metric ball of radius t about ξ associated to p.

Proof. Let p_t parameterize the geodesic ray through O and asymptotic to p with $p_0 = O$, $p_{-\infty} = O_\partial$ and $p_{+\infty} = p$. For $R > r$, let σ_R be a loxodromic isometry of $\mathcal{H}_{\mathbb{F}}^n$ fixing p and O_∂ and translating p_R to p_r. Then $\sigma_R S_r(O, p, R) = S_r(p_{-R-r}, p, R)$ is the shadow from p_{-R-r} of the ball of radius r centered at p_r. First, we see $S_r(O, p, r)^c \subset \sigma_R S_r(O, p, R)^c$ so since $S_r(O, p, r)^c$ is an open neighborhood of O_∂, we can find A_1 such that

$$B(O_\partial, A_1) \subset S_r(O, p, r)^c \subset \sigma_R S_r(O, p, R)^c$$

for all $R > r$. As R goes to ∞, $\sigma_R S_r(O, p, R)$ shrinks to the shadow S from O_∂ of the ball of radius r centered at p_r, whose complement is a bounded neighborhood of O_∂ containing $S_r(p_{R-r}, p, R)^c$ for all $R > r$, and hence we can find an A_2 such that

$$\sigma_R S_r(O, p, R)^c \subset S^c \subset B(O_\partial, A_2).$$

Then we have

$$\sigma_R^{-1} B(O_\partial, A_1) \subset S_r(O, p, R) \subset S^c \subset \sigma_R^{-1} B(O_\partial, A_2).$$

Since σ_R^{-1} is the element of the isometry group A corresponding to e^R in the identification $A \cong \mathbb{R}^*$, using Equation (6) we have that σ_R^{-1} scales the metric by e^R. Thus

$$\sigma_R^{-1} B(O_\partial, C) = B(O_\partial, C e^R),$$

and the result follows. ☐

3.2. **Choosing standard horoballs.** Let $O \in \mathcal{H}_{\mathbb{F}}^n$ be a fixed reference point and Γ a non-elementary geometrically finite group. Let F be a convex fundamental domain for Γ containing O in its interior. Then F has a finite number of parabolic cusps, and hence a finite set $P \in \partial\mathcal{H}_{\mathbb{F}}^n$ of associated parabolic fixed points. The set of all parabolic fixed points of Γ is then the orbit ΓP. Let $p \in P$, and let Γ_p be the stabilizer of p in Γ. Let Q_p be the associated tile of diameter q_p as defined in Section 2.4. We call the collection $\{\gamma Q_p : \gamma \in \Gamma_p\}$ the set of "tiles."

By Margulis' Lemma (Theorem 2.4), we may choose a collection $\{\mathcal{H}_{\gamma p} : \gamma p \in \Gamma P\}$ of disjoint standard horoballs. The Global Measure Formula deals with the measure of shadows on $\partial\mathcal{H}_{\mathbb{F}}^n$ of balls whose center lies in the thin part of M, i.e. in one of the standard horoballs. We will need to make this choice in such a way that O is in the thick part of M_Γ and that the thin part of M_Γ is sufficiently far away from O.

This technical assumption will insure that shadows of balls centered at points of $\mathcal{H}_p$ will be large enough to be approximated using the sets $\Gamma_p Q_p$. Vaguely, we will want to estimate the measure by counting how many tiles fit inside; if the shadow were too small, there would be no tiles inside, and we would not have a lower bound for the measure. We will use this assumption in Proposition 4.2.

For each $p \in P$, Proposition 3.2 gives us a constant A_1 associated to p; for each $p \in P$, let A_p denote this constant. Now let $a = \min\{A_p : p \in P\}$. For $q = \max\{q_p : p \in P\}$, we choose $\mathcal{H}_p$ such that the distance from O to $\mathcal{H}_p$ satisfies

$$ae^{\rho(O,\mathcal{H}_p)} > 3q. \tag{7}$$

This defines $\mathcal{H}_p$ for each $p \in P$, and we will prove the Global measure formula for shadows of balls centered at points of this finite collection of horoballs in Section 4. For $q = \gamma p \in \Gamma P$, let $\mathcal{H}_q = \gamma\mathcal{H}_p$.

We will need to shrink these again before completing the argument; we make the final choice of horoballs $\mathcal{H}'_q$ for which we prove the final result in Section 5.

3.3. **On the sizes and numbers of tiles.** Let $p \in P$, with stabilizer $\Gamma_p \subseteq \Gamma$, and let Q be the tile constructed in Section 2.5 associated to p, and let q denote its diameter. Let H, Γ_1 and Γ_2 be the groups given in Theorem 2.9. We will estimate the measures of our shadows by estimating the number and measures of the tiles that fit within them. The following lemma tells us that for $g \in \Gamma_p$ to estimate the μ_O measure of the tile gQ, we must estimate $\rho(O, gO)$.

Lemma 3.4. *There exist constants C and C' such that if $g \in \Gamma_p$, then*

$$C' \leq \frac{\mu_O(gQ)}{e^{-\delta\rho(O,gO)}} \leq C.$$

Proof. Using the properties of the Patterson Sullivan measures (Proposition 2.11), for $g \in \Gamma_p$ we have

$$\mu_O(gQ) = \mu_{g^{-1}O}(Q) = \int_{\eta\in Q} e^{-\delta b^O_\eta(g^{-1}O)} d\mu_O[\eta]. \tag{8}$$

Recall that we chose Q to be relatively compact and containing the endpoint O_∂ of the geodesic from p to O. Let $HB_\eta(O)$ denote the horoball whose bounding horosphere is centered at η and passes through O. Let K' be the union over all $\eta \in Q$ of the horoballs $HB_\eta(O)$. Then the set $K = H_p(O) \cap K'$ is a compact neighborhood of O on $H_p(O)$ such that for $y \in H_p(O)\backslash K$ and $\eta \in Q$, we have $b^O_\eta(y) > 0$.

Note the orbit $\Gamma_p O \subseteq H_p(O)$. Since Γ_p is discrete, at most a finite number of orbit points from $\Gamma_p O$ lie in K, and hence there exists an $\epsilon > 0$ such that for all but a finite number of $g \in \Gamma_p$, we have $b^O_\eta(gO) > \epsilon$. Call the subset of Γ_p omitting this finite set by G.

For $g \in G$, let $X_g \in \mathcal{H}^n_{\mathbb{F}}$ be the point at which the geodesic asymptotic to η and through O intersects $H_\eta(gO)$. We apply Lemma 2.1 to the triangle with vertices O, X_g and gO. The side $\overline{OX_g}$ has length $b^O_\eta(gO) > \epsilon$, and since O, gO and X_g do not lie on a single geodesic, the angle of our triangle across from the side $\overline{OX_g}$ is bounded below for all $\eta \in Q$ and all $g \in G$. By Lemma 2.1, we have a constant C such that

$$d(O, gO) - C \leq d(O, gO) + d(gO, X) - C \leq b^O_\eta(gO) \leq d(O, gO),$$

which implies

$$e^{-d(O,gO)} \leq e^{-b^O_\eta(gO)} \leq e^C e^{-d(O,gO)},$$

for all $\eta \in Q$, and by adjusting the constant, for all $g \in \Gamma_p$.

Combining this with Equation (8), we have

$$\mu_O(Q)e^{-\delta d(O,gO)} \le \int_{\eta\in Q} e^{-\delta b^O_\eta(g^{-1}O)} d\mu_O[\eta] \le e^C \mu_O(Q)e^{-\delta d(O,gO)},$$

and $e^{-\delta d(O,gO)} \asymp \mu_O(gQ)$, as desired. ☐

Let $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and $r \in \mathbb{R}^+$. Let $B(\xi, r)$ denote the ball about ξ of radius r with respect to the Cygan metric associated to p. Since the support of the Patterson-Sullivan measure μ_O lies in the orbit $\Gamma_p Q$, we study $\mu_O(B(\xi, r))$ for $\xi \in \Lambda_\Gamma\backslash\{p\}$ by considering the intersection of the orbit $\Gamma_p Q$ with the ball $B(\xi, r)$. In the following Lemma, we prove that the number of tiles gQ (for $g \in \Gamma_p$) that hit a ball of radius r has bounds that are proportional to a power of r, where the power is related to the rank (k, l) of p. We want to take advantage of the structure of the Lie subgroup H. Since q bounds the Cygan distance from $\Lambda_\Gamma\backslash\{p\}$ to H, the assumption $r > 3q$ in the following Lemma implies that a ball of radius r centered at a point of $\Lambda_\Gamma\backslash\{p\}$ will sufficiently intersect H. Let $\#A$ denote the cardinality of a set A.

Lemma 3.5. *For $\xi \in \Lambda_\Gamma\backslash\{p\}$, let $S(\xi, r) = \{g \in \Gamma_p : gQ \cap B(\xi, r) \ne \emptyset\}$. Then there are constants C and C' such that for $r > 3q$,*

$$C \le \frac{\#S(\xi, r)}{r^{k+2l}} \le C',$$

for all $\xi \in \Lambda_\Gamma\backslash\{p\}$.

Proof. The Riemannian metric on $\mathcal{H}^n_{\mathbb{F}}$ induces a Riemannian metric on $H_p(O)$. Let Vol_p denote the push forward to the boundary of the volume form related to the Riemannian metric on $H_p(O)$ under the projection $\pi_p : H_p(O) \to \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ along geodesics asymptotic to p.

For $\xi \in \Lambda_\Gamma\backslash\{p\}$, let ξ_H be the closest point to ξ on H. Since $d_p(\xi, \xi_H) < q$ and $r > 3q$, judicious use of the triangle inequality shows

$$B(\xi_H, \frac{2}{3}r) \subseteq B(\xi, r) \subseteq B(\xi_H, \frac{4}{3}r).$$

Since Γ_1 is of finite index in Γ_p, and since $Q \cap H$ is a fundamental domain for the action of Γ_1, there are constants C and C' such that the volume $\mathrm{Vol}_p(B(\xi, r) \cap \Gamma_p(Q))$ is bounded above and below proportional to volumes of balls in H:

$$C'\mathrm{Vol}_p\big|_H \left(B(\xi_H, \frac{2}{3}r)\right) \le \mathrm{Vol}_p\left(B(\xi, r) \cap \Gamma_p Q\right) \le C\mathrm{Vol}_p\big|_H \left(B(\xi_H, \frac{4}{3}r)\right).$$

Note that the constants C and C' depend only on Γ_p. Since elements of Γ_p preserve Vol_p, we have $\mathrm{Vol}_p(gQ) = \mathrm{Vol}_p(Q)$. Thus

$$\#S(\xi, r) \asymp \frac{\mathrm{Vol}_p\big|_H(B(\xi_H, r) \cap H)}{\mathrm{Vol}_p(Q)},$$

and we calculate $\mathrm{Vol}_p\big|_H(B(\xi_H, r) \cap H)$.

We will use Heisenberg coordinates for elements of $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and the Cygan metric to describe $B(\xi_H, R)$. By definition (Equation 4),

$$B((0, 0), R) = \left\{(\zeta, v) \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\} : |\zeta|^4 + |v|^2 < R^4\right\}.$$

The volume Vol_p is invariant under Heisenberg translations (i.e. the action of N), and if we translate by an element of H, the volume restricted to H, $\mathrm{Vol}_p\big|_H$, will likewise be preserved. Thus we have $\mathrm{Vol}_p\big|_H B(\xi_H, R) = \mathrm{Vol}_p\big|_H B((0, 0), R)$.

Now we calculate $\mathrm{Vol}_p\big|_H B((0,0),R)$, noting that in Heisenberg coordinates, $d\mathrm{Vol}_p\big|_H = |\zeta|^{k-1}|v|^{l-1}d|\zeta|d|v|$.

$$\begin{aligned}
\mathrm{Vol}_p\big|_H B((0,0),R) &= \int_{B(\xi_H,R)} |\zeta|^{k-1}|v|^{l-1}d|\zeta|d|v| \\
&= 4\int_0^{R^2}\int_0^{\sqrt[4]{R^4-x^2}} x^{l-1}y^{k-1}dydx \\
&= \frac{4}{k}\int_0^{R^2}(R^4-x^2)^{\frac{k}{4}}x^{l-1}dx \\
&= \frac{4}{k}\int_0^{\frac{\pi}{2}} R^{k+2l}(1-\sin^2\theta)^{\frac{k}{4}}\sin^{l-1}\theta\cos\theta d\theta \\
&\asymp R^{k+2l},
\end{aligned}$$

using the trigonometric substitution $x = R^2\sin\theta$. Thus we have

$$\mathrm{Vol}_p\big|_H(B(\xi_H,r)\cap H) \asymp r^{k+2l},$$

and hence

$$\#S(\xi,r) \asymp r^{k+2l}.$$

as desired. ☐

This idea also allows us to count the number of tiles in an annulus.

Corollary 3.6. *There exist constants C and C' such that for all $r > 0$,*

$$C' \leq \frac{\#\{g\in\Gamma_p : (B(O_\partial, r+q_o)\backslash B(O_\partial,r))\cap gQ \neq \emptyset\}}{r^{k+2l-1}} \leq C.$$

Proof. This follows since the volume $Vol_p((B(O_\partial,r-q_o)\backslash B(O_\partial,r))\cap\Gamma_p Q)$ is proportional to the surface area restricted to H of a ball of radius r. ☐

4. Proof of the Global Measure Formula when the shadows are of balls centered at points of $\mathcal{H}_p$ for $p \in P$

4.1. **Cases.** Now we assume that we have a collection of standard horoballs $\{\mathcal{H}_{\gamma p} : p \in P, \gamma\in\Gamma\}$, with $\mathcal{H}_p$ satisfying Equation (7) for each $p\in P$. We fix an element $p \in P$ for all of Section 4, and consider the shadow $S(O,\xi,R)$ when $\xi_R \in \mathcal{H}_p$. We will prove that there exist constants C and C' such that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and $R > 1$ satisfying $\xi_R \in \mathcal{H}_p$, we have

$$C \leq \frac{\mu_O(S(O,\xi,R))}{e^{-\delta R}\left(e^{\rho(\xi_R,\Gamma O)}\right)^{rank(\xi_R)-\delta}} \leq C'.$$

Note that in this case, $rank(\xi_R) = 2l+k$, where (l,k) is the rank of p.

Here is the plan for the proof of the Global Measure Formula in this case. Let $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ such that the geodesic connecting O to ξ passes through $\mathcal{H}_p$. Then for R near 1, the shadow $S(O,\xi,R)$ will contain p. If $\xi \neq p$, eventually, as R increases, the shadow will shrink toward ξ, and p will no longer lie inside it. We will consider three cases: when p lies well outside the shadow, well inside the shadow and near the boundary of the shadow. In fact, we will prove the result for shadows for which p lies away from the boundary in Sections 4.2 and 4.3, and combine them in Section 4.4 to acquire the third case.

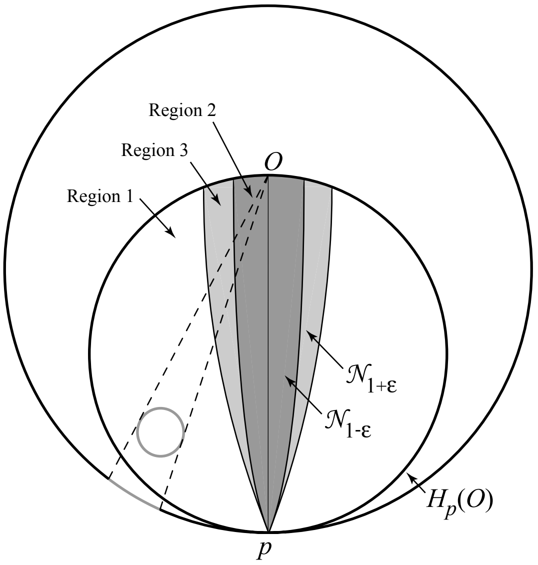

Figure 3. The point p lies well outside shadows of balls centered in Region 1, well inside shadows of balls centered in Region 2 and near the boundary of balls centered in Region 3.

We will use the following notation. See Figure 3. Fix $0 < \epsilon < 1$, and let $\overline{Op}$ denote the geodesic arc connecting O to p, and let $\mathcal{N}_{1-\epsilon}$ and $\mathcal{N}_{1+\epsilon}$ denote $1-\epsilon$ and $1+\epsilon$ neighborhoods of $\overline{Op}$ respectively. Then our three cases are as follows. For $\xi_R \in \mathcal{N}_{1-\epsilon}$, p lies inside the shadow $S(O, \xi, R)$, since the ball of radius 1 about ξ_R intersects $\overline{Op}$, and for $\xi_R \in \mathcal{N}^c_{1+\epsilon}$, p lies outside the closure of the shadow $S(O, \xi, R)$, since the closed ball of radius 1 about ξ_R does not intersect $\overline{Op}$. For $\xi_R \in \mathcal{N}_{1+\epsilon} \backslash \mathcal{N}_{1-\epsilon}$, p lies near the boundary of $S(O, \xi, R)$.

Note that in the statement of the Global Measure Formula, the rate of growth of the measure of a shadow centered at the point ξ_R is given in terms of the distance $d(\xi_R, \Gamma O)$. In fact we will prove the statement with $d(\xi_R, \Gamma O)$ replaced by $-b^O_p(\xi_R)$, which we can do since there exists a constant C such that for any $\xi \in \Lambda_\Gamma$ and $R > 0$ such that $\xi_R \in H_p(O)$, we have

$$\rho(\xi_R, \Gamma O) - C \leq -b^O_p(\xi_R) \leq \rho(\xi_R, \Gamma O) + C. \tag{9}$$

We briefly explain how this follows from the geometric finiteness of Γ. Recall that we chose F to be the fundamental domain for Γ consisting of the points in $\mathcal{H}^n_{\mathbb{F}}$ that lie closer to O than to any other point of ΓO, and that p is a vertex at infinity of that region. Without loss of generality, assume $O \in Hull(\Lambda_\Gamma)$. Then $\xi_R \in Hull(\Lambda_\Gamma)$ for all R, $\xi \in \Lambda_\Gamma$. Let σ_ξ denote the part of the geodesic ray connecting O to ξ that lies in the horoball bounded by $H_p(O)$. And let $K = \bigcup_{\xi \in \Lambda_\Gamma} \sigma_\xi$. Then since Γ is geometrically finite and $K \subseteq Hull(\Lambda_\Gamma)$, the quotient of any ϵ neighborhood of K by Γ has finite volume. Since K lives in the horoball bounded by $H_p(O)$, we can project it out to a set $K' \subseteq H_p(O)$ along geodesics asymptotic to p. K' may not be compact in $H_p(O)$. However in the quotient, K'/Γ will lie a compact subset of the set $H_p(O)/\Gamma$ that bounds the cusp at p. Let C be the diameter of this convex set. Then K' will lie in the union of balls of radius C about the orbit points of O. Now consider $\xi \in \Lambda_\Gamma$ and R such that $\xi_R \in H_p(O)$. Let $\gamma \in \Gamma$ such that $\rho(\xi_R, \Gamma O) = \rho(\xi_R, \gamma O)$. Let x_ξ be the intersection of the geodesic from p through ξ_R with the horosphere $H_p(O)$. Then x_ξ lies in K' and hence lies within C of γO. Since $\rho(\xi_R, x_\xi) = -b_p^O(\xi_R)$, by the triangle inequality, we have

$$\rho(\xi_R, \gamma O) - C \leq -b_p^O(\xi_R) \leq \rho(\xi_R, \gamma O) + C.$$

as desired.

4.2. When p is not in the shadow $S(O, \xi, R)$.

Our plan in this section and the next is to estimate the measure of the shadow by estimating the number and measure of tiles gQ (for $g \in \Gamma_p$) that hit the shadow. In light of Lemma 3.4, to study the measure of tiles gQ in our shadow, we study $\rho(O, gO)$.

Recall that π_p is the map that identifies $H_p(O)$ to $\partial\mathcal{H}^n_{\mathbb{F}}$ along geodesics asymptotic to p. Thus as Γ_p fixes $H_p(O)$, for $g \in \Gamma_p$, we have $\pi_p^{-1} gO_\partial = gO$. The tiles we are interested in are those gQ such that $gQ \cap S(O, \xi, R) \neq \emptyset$. Since when $gQ \cap S(O, \xi, R) \neq \emptyset$, gO is essentially in $\pi_p^{-1} S(O, \xi, R)$, we will estimate $\rho(O, \pi_p^{-1} S(O, \xi, R))$. See Figure 4.

Note that for $x \in \mathcal{H}^n_{\mathbb{F}}$, $b_p^O(x)$ measures the (signed) distance between the horosphere based at p and through O and the horosphere based at p and through x. The following lemma gives an estimate for $\rho(\pi_p^{-1} z, O)$, for z in the shadow $S(O, \xi, R)$, in terms of $d = -b_p^O(\xi_R)$.

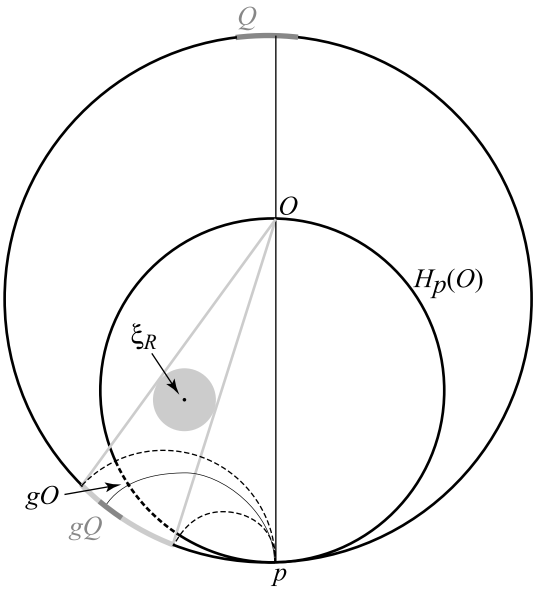

FIGURE 4. This figure shows gO when gQ lies in $S(O, \xi, R)$. The dotted line on $H_p(O)$ is $\pi_p^{-1} S(O, \xi, R)$.

Lemma 4.1. *There exist constants C and C' such that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and $R > 1$ satisfying $\xi_R \in \mathcal{N}^c_{1+\epsilon} \cap \mathcal{H}_p$, we have*

$$C' \leq \frac{e^{\rho(\pi_p^{-1} z, O)}}{e^{d+R}} \leq C,$$

for all $z \in S(O, \xi, R)$, where $d = -b_p^O(\xi_R) > 0$.

Proof. Let $z = [Z] \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$, with $Z = (Z', Z_n, 1) \in \mathbb{F}^{n,1}$ where $Z' \in \mathbb{F}^{n-1}$ and $Z_n \in \mathbb{F}$. Then by Claim 2.2,

$$\pi_p^{-1}(z) = \left[\begin{pmatrix} \frac{Z'}{Z_n+1} \\ \frac{1}{2}\left(\frac{Z_n-1}{Z_n+1}\right) \\ \frac{1}{2}\left(2 - \frac{Z_n-1}{Z_n+1}\right) \end{pmatrix}\right],$$

and we have

$$e^{\frac{\rho(O, \pi_p^{-1}(z))}{2}} \asymp \cosh\frac{\rho(O, \pi_p^{-1}(z))}{2} = \frac{1}{2}\left|\frac{2}{Z_n+1} + 1\right| \asymp \frac{1}{|Z_n+1|}.$$

The geodesic ray beginning at O and asymptotic to z can be parameterized by

$$z_t = \left[\begin{pmatrix} Z' \tanh\frac{t}{2} \\ Z_n \tanh\frac{t}{2} \\ 1 \end{pmatrix}\right],$$

which when $t \neq 0$ is equal to

$$\left[\begin{pmatrix} Z' \\ Z_n \\ \frac{1}{\tanh\frac{t}{2}} \end{pmatrix}\right].$$

Note that using the change of coordinates in Equation (3), in horospherical coordinates, the $\mathbb{R}^+$ coordinate of z_t will be

$$-Re\left(\frac{Z_n - \frac{1}{\tanh\frac{t}{2}}}{Z_n + \frac{1}{\tanh\frac{t}{2}}}\right) - \frac{\ll Z', Z' \gg}{|Z_n + \frac{1}{\tanh\frac{t}{2}}|^2} = \frac{\frac{4e^t}{(e^t-1)^2}}{|Z_n + \frac{1}{\tanh\frac{t}{2}}|^2}. \tag{10}$$

Now since z_R is within 1 of ξ_R, $b_p^O(z_R)$ is within 1 of $-d = b_p^O(\xi_R)$. Thus we have

$$e^d \asymp \frac{\frac{4e^R}{(e^R-1)^2}}{|Z_n + \frac{1}{\tanh\frac{R}{2}}|^2}. \tag{11}$$

Using Equation (11) and the triangle inequality for $|\cdot|$ in $\mathbb{F}$, we get

$$\frac{1}{|Z_n+1|} \asymp e^{\frac{d+R}{2}}$$

as desired. □

Now using Proposition 3.2 to estimate our shadows in terms of Cygan balls, and Lemma 3.5 to estimate the number of tiles that may land in a ball, and Lemma 4.1 above to estimate the measures of these tiles, we have the following proposition, proving the Global Measure Formula in this case.

Proposition 4.2. *There exist constants C and C' such that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and $R > 1$ satisfying $\xi_R \in \mathcal{N}^c_{1+\epsilon} \cap \mathcal{H}_p$, we have*

$$C \leq \frac{\mu_O(S(O,\xi,R))}{e^{-\delta R}\big(e^{\rho(\xi_R,\Gamma O)}\big)^{rank(\xi_R)-\delta}} \leq C'.$$

Proof. Let $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and $R > 1$ satisfy $\xi_R \in \mathcal{N}^c_{1+\epsilon}$, and let $B(\xi,t)$ be a ball with respect to the Carnot-Caratheodory or Cygan metric about $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and of radius t. By Proposition 3.2, there are positive constants A_1 and A_2 independent of $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and R such that,

$$B(\xi, A_1e^d) \subset S_r(O,\xi,R) \subset B(\xi, A_2e^d),$$

By Lemma 3.5, and our choice of standard horoballs in Section 3.2 (insuring $A_1e^d > 3q$), the number of $g \in \Gamma_p$ such that $gQ \cap S(O,\xi,R) \neq \emptyset$ is bounded between $(A_1e^d)^{k+2l}$ and $(A_2e^d)^{k+2l}$. By Lemma 3.4, to estimate the μ_O measure of a tile gQ, we must estimate $\rho(O,gO)$. Let $gQ \cap S(O,\xi,R) \neq \emptyset$. Then gO is in $\pi_p^{-1}S(O,\xi,R)$, and we apply Lemma 4.1, to get $\mu_O(gQ) \asymp e^{-\delta(d+R)}$. Thus we have

$$\mu_O(S(O,\xi,R)) \asymp e^{-\delta(d+R)}e^{d(k+2l)}$$

Recall that $rank(\xi_R) = k + 2l$ for $\xi_R \in \mathcal{H}_p$, where p has rank (k,l). In light of this and Equation (9), we put our result in the desired form. □

4.3. **When p is in the shadow $S(O,\xi,R)$.** Since in the case $\xi_R \in \mathcal{N}_{1-\epsilon}$, p lies well within the shadow $S(O,\xi,R)$, we bound our shadow between shadows of balls centered at points of the geodesic $\overline{Op}$. The first two lemmas explain that this can be done in a uniform manner, so our result will be independent of ξ.

Lemma 4.3. *There is a constant C such that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and $t > 0$ such that $\xi_t \in \mathcal{N}_{1-\epsilon}$,*

$$S_\epsilon(O,p,t) \subseteq S(O,\xi,t) \subseteq S_{2-\epsilon}(O,p,t-C).$$

Proof. For a given $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and $t > 0$, let t' be such that $p_{t'}$ is the closest point to ξ_t on the the geodesic $\overline{Op}$. Then for ξ and t such that $\xi_t \in \mathcal{N}_{1-\epsilon}$, we have

$$B(p_{t'},\epsilon) \subseteq B(\xi_t,1) \subseteq B(p_{t'},2-\epsilon),$$

by the triangle inequality where $B(x,r)$ denotes the ball of radius r about x with respect to ρ. Note that the difference $t - t'$ between t and t' is bounded for all ξ, i.e. there exists a constant C such that for all ξ and t such that $\xi_t \in \mathcal{N}_{1-\epsilon}$, we have $t - C < t' < t$. So

$$S_\epsilon(O,p,t) \subseteq S_\epsilon(O,p,t') \subseteq S(O,\xi,t) \subseteq S_{2-\epsilon}(O,p,t') \subseteq S_{2-\epsilon}(O,p,t-C).$$

□

Now we will study shadows $S_r(O,p,R)$ of balls of radius r centered at points on the geodesic $\overline{Op}$. By Proposition 3.3, we have constants A_1 and A_2 such that

$$B(O_\partial, A_2e^R)^c \subseteq S_r(O,p,R) \subseteq B(O_\partial, A_1e^R)^c,$$

where $B(\xi,t)$ denotes the Cygan metric ball of radius t about ξ. So we calculate the measure of the complement of Cygan balls centered at O_∂. Since the complement of such a ball is of infinite volume, the volume argument we used when p was not in the shadow will not work without modification. Our solution is to exhaust these complements with finite volume annuli, and, on each of these, use a similar approach to that of the previous case.

Recall that q is the diameter of the tile Q in the Cygan metric. Let $n \in \mathbb{Z}$, and let $Strip(n) = B(O_\partial, nq) \backslash B(O_\partial, (n-1)q)$. We bound $Strip(n)$ between two shadows. Using Proposition 3.3, we have constants B_1, B_2 such that if

$$T_i(n) = \ln\left(\frac{nq}{B_i}\right),$$

for $i = 1, 2$, then

$$S(O, p, T_2(n))^c \subseteq B(O_\partial, nq) \subseteq S(O, p, T_1(n))^c.$$

So $Strip(n) \subseteq S(O, p, T_2(n-1)) \backslash S(O, p, T_1(n))$. Note that $T_2(n-1)$ and $T_1(n)$ differ by at most $c = \ln(\frac{B_2}{B_1})$. The following lemma and its corollary provide an estimate for the measure of a tile that hits the n^{th} strip.

Lemma 4.4. *There exist constants C and C' such that if $g \in \Gamma_p$ such that $gQ \cap Strip(n) \neq \emptyset$, then*

$$C' \leq \frac{\mu_O(gQ)}{n^{-2\delta}} \leq C.$$

Proof. If $gQ \cap Strip(n) \neq \emptyset$, then gO is essentially in $\pi_p^{-1} Strip(n)$, where again, π_p is the projection along geodesics asymptotic to p from $H_p(O)$ onto $\partial\mathcal{H}^n_{\mathbb{F}}$. Now $Strip(n) \subseteq S(O, p, T) \backslash S(O, p, T + c)$, where $c = \ln(\frac{B_2}{B_1})$, and $T = T_2(n-1)$. By Lemma 3.4, we have $\mu_O(gQ) \asymp e^{-\delta\rho(O,gO)}$, and hence we approximate $e^{-\delta\rho(O,\pi_p^{-1}\omega)}$ for $\omega \in S(O, p, T) \backslash S(O, p, T + c)$.

We will show there exist constants C and C' such that if $T > 1$ and $\omega \in S(O, p, T) \backslash S(O, p, T + c)$,

$$C' \leq \frac{e^{\rho(O,\pi_p^{-1}w)}}{e^{2T}} \leq C.$$

Since $\omega \in S(O, p, T) \backslash S(O, p, T + c)$, the distance from the geodesic connecting O to p to ω_T is less than 1, but the distance from the geodesic connecting O to p to ω_{T+c} is greater than 1. Thus $\rho(\omega_{T+c}, p_{T+c}) < 1$ and $\rho(\omega_T, p_T) > 1$.

Let $\omega = [(w', w_n, 1)]$, where $w' \in \mathbb{F}^{n-1}$, $w_n \in \mathbb{F}$. Since $\omega \in \partial\mathcal{H}^n_{\mathbb{F}} \backslash \{p\}$, we have $\ll w', w' \gg + |w_n|^2 = 1$. Then we have

$$\omega_t = \left[\begin{pmatrix} w' \\ w_n \\ \frac{1}{\tanh\frac{t}{2}} \end{pmatrix}\right].$$

Then using Equation (1), we have

$$\cosh\frac{\rho(\omega_T, p_T)}{2} = \frac{\left|w_n + \frac{1}{\tanh^2\frac{T}{2}}\right|}{\left|1 - \frac{1}{\tanh^2\frac{T}{2}}\right|} < \cosh\frac{1}{2},$$

and

$$\cosh\frac{\rho(\omega_{T+c}, p_{T+c})}{2} = \frac{\left|w_n + \frac{1}{\tanh^2\frac{T+c}{2}}\right|}{\left|1 - \frac{1}{\tanh^2\frac{T+c}{2}}\right|} > \cosh\frac{1}{2}.$$

Simplifying by combining these and using the triangle inequality, we get

$$\frac{1}{|1 + w_n|} \asymp e^T.$$

Since by Claim 2.2, we have

$$e^{\frac{\rho(O,\pi_p^{-1}w)}{2}} \asymp \frac{1}{|1+w_n|},$$

we have

$$e^{\rho(O,\pi_p^{-1}w)} \asymp e^{2T},$$

as desired.

Now we have $e^{\rho(O,gO)} \asymp e^{2T_2(n-1)} \asymp n^2$ by the definition of $T_1(n)$. Thus we have $\mu_O(gQ) \asymp n^{-2\delta}$, for all g such that $gQ \cap Strip(n) \neq \emptyset$, completing the proof. ☐

The approach for the proof of Proposition 4.5 follows an idea from [29]. This completes the argument in this case.

Proposition 4.5. *There exist constants C and C' such that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and $R > 1$ satisfying $\xi_R \in \mathcal{N}_{1-\epsilon} \cap \mathcal{H}_p$, we have*

$$C \leq \frac{\mu_O(S(O,\xi,R))}{e^{-\delta R}\big(e^{\rho(\xi_R,\Gamma O)}\big)^{rank(\xi_R)-\delta}} \leq C'.$$

Proof. Since by Lemma 4.3, there is a constant c such that for all $\xi \in \partial\mathcal{H}^n_{\mathbb{F}}$ and $t > 0$ such that $\xi_t \in \mathcal{N}_{1-\epsilon}$,

$$S_\epsilon(O,p,t) \subseteq S(O,\xi,t) \subseteq S_{2-\epsilon}(O,p,t-c), \tag{12}$$

we study the measure of a shadow $S(O,p,R)$, centered at p.

First we will show for each $r > 0$, there exist constants C and C' such that for all $R > r$,

$$C' \leq \frac{\mu_O(S_r(O,p,R))}{e^{-2\delta R}e^{(k+2l)R}} \leq C,$$

by bounding $S_r(O,p,R)$ in terms of the strips $Strip(n)$ and estimating $\mu_O(Strip(n))$.

We have

$$B(O_\partial, A_2e^R)^c \subseteq S_r(O,p,R) \subseteq B(O_\partial, A_1e^R)^c,$$

so

$$\mu_O(B(O_\partial, A_2e^R)^c) \leq \mu_O(S_r(O,p,R)) \leq \mu_O(B(O_\partial, A_1e^R)^c).$$

Since

$$\bigcup_{n\geq t} Strip(n) \subseteq B(O_\partial, t)^c \subseteq \bigcup_{n\geq t-1} Strip(n),$$

we have

$$\sum_{n\geq A_2e^R} \mu_O(Strip(n)) \leq \mu_O(S_r(O,p,R)) \leq \sum_{n\geq A_1e^R+1} \mu_O(Strip(n))$$

Now by Corollary 3.6,

$$\begin{aligned} &\#\{g \in \Gamma_p : Strip(n) \cap gQ \neq \emptyset\} \\ &\quad = \#\{g \in \Gamma_p : (B(O_\partial, nQ)\backslash B(O_\partial,(n-1)Q)) \cap gQ \neq \emptyset\} \asymp n^{k+2l-1}. \end{aligned} \tag{13}$$

By Lemma 4.4, if $g \in \Gamma_p$ such that $gQ \cap Strip(n) \neq \emptyset$, then $\mu_O(gQ) \asymp n^{-2\delta}$. So $\mu_O(Strip(n)) \asymp n^{-(2\delta-(k+2l-1))}$.

By a result from number theory, there exist positive constants c and c' such that

$$ct^{1-s} \leq \sum_{n\geq t} n^{-s} \leq c't^{1-s}.$$

Thus

$$\sum_{n\geq t}\mu_O(Strip(n))\asymp\sum_{n\geq t}n^{-(2\delta-(k+2l)+1)}\asymp t^{-2\delta+(k+2l)},$$

and evaluating for $t=A_1e^R+1$ and $t=A_2e^R$, for each $r>0$ we get

$$\mu_O(S_r(O,p,R))\asymp e^{-2\delta R}e^{(k+2l)R},$$

and applying Equation (12), we have

$$\mu_O(S(O,\xi,R))\asymp e^{-2\delta R}e^{(k+2l)R},$$

as well.

Note that when $\xi_R\in\mathcal{N}_{1-\epsilon}$, the horospheric distance $d=-b_p^O(\xi_R)$ from ξ_R to O is within $1-\epsilon$ of R; in light of Equation (9), we have

$$e^R\asymp e^{\rho(\xi_R,\Gamma O)},$$

and the result follows. □

4.4. **When p is near the edge of the shadow.** In this section we use that the result holds for $\xi\in\partial\mathcal{H}^n_{\mathbb{F}}$ and $R>1$ with $\xi_R\in(\mathcal{N}_{1-\epsilon}\cup\mathcal{N}^c_{1+\epsilon})\cap\mathcal{H}_p$ to show it holds for ξ_R in the gap $\mathcal{N}^c_{1-\epsilon}\cap\mathcal{N}_{1+\epsilon}$. First we show that there is a constant c such that for all ξ, the geodesic from O to ξ remains in the gap for at most length c.

Lemma 4.6. *There exists a constant $c>0$, such that for all $\xi\in\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ the interval $I_\xi=\{t\in\mathbb{R}:\xi_t\in\mathcal{N}_{1+\epsilon}\backslash\mathcal{N}_{1-\epsilon}\}$ has length less than c.*

Proof. Let $\overline{OP}$ denote the geodesic through O and asymptotic to p. For any compact set of $\xi\in\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$, we can find such a constant, so we consider a sequence $\xi^i\to p$ in $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$. Let t_i be the time at which the geodesic from O to ξ^i enters $\mathcal{N}_{1+\epsilon}\backslash\mathcal{N}_{1-\epsilon}$ (i.e. such that $d(\xi^i_{t_i},\overline{OP})=1-\epsilon$). Let t'_i be the time at which the geodesic from O to ξ exits $\mathcal{N}_{1+\epsilon}\backslash\mathcal{N}_{1-\epsilon}$ (i.e. such that $d(\xi^i_{t'_i},\overline{OP})=1+\epsilon$). Then we want $d(\xi^i_{t_i},\xi^i_{t'_i})=t'_i-t_i$ bounded for all i.

Now the loxodromic isometries fixing the geodesic $\overline{OP}$ also preserve the boundaries of $\mathcal{N}_{1+\epsilon}$ and $\mathcal{N}_{1-\epsilon}$. Let σ_i be such a loxodromic isometry taking $\xi^i_{t_i}$ to $\xi^1_{t_1}$. Then $\sigma_i(O)$ tends toward $O_\partial\in\partial\mathcal{H}^n_{\mathbb{F}}$, and $\sigma_i(\xi^i_{t'_i})$ tends to the point where the geodesic connecting O_∂ to $\xi^1_{t_1}$ intersects $\partial\mathcal{N}_{1+\epsilon}$, a bounded distance away from $\xi^1_{t_1}=\sigma_i(\xi^i_{t_i})$. Thus $d(\sigma_i(\xi^i_{t_i}),\sigma_i(\xi^i_{t'_i}))=d(\xi^i_{t_i},\xi^i_{t'_i})=t_i-t'_i$ is bounded for all i, as desired. □

Interpolating between Proposition 4.2 and Proposition 4.5, we get the following proposition, completing our work for $\xi_R\in\mathcal{H}_p$.

Proposition 4.7. *There exist constants C and C' such that for all $\xi\in\partial\mathcal{H}^n_{\mathbb{F}}$ and $R>1$ satisfying $\xi_R\in\mathcal{N}_{1+\epsilon}\cap\mathcal{N}^c_{1-\epsilon}\cap\mathcal{H}_p$, we have*

$$C\leq\frac{\mu_O(S(O,\xi,R))}{e^{-\delta R}\big(e^{\rho(\xi_R,\Gamma O)}\big)^{rank(\xi_R)-\delta}}\leq C'.$$

Proof. Let $x\in\mathcal{N}_{1+\epsilon}\cap\mathcal{N}^c_{1-\epsilon}\cap\mathcal{H}_p$. By Lemma 4.6, there exists a $c>0$ such that translating x back along the geodesic connecting it to O by c puts x in our first case, and forward by c puts x in our second case. In other words, if $\xi\in\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ and $R>1$ satisfying $\xi_R\in\mathcal{N}_{1+\epsilon}\cap\mathcal{N}^c_{1-\epsilon}\cap\mathcal{H}_p$, then $\xi_{R-c}\in\mathcal{N}_{1-\epsilon}$ and $\xi_{R+c}\in\mathcal{N}^c_{1+\epsilon}$. See Figure 5.

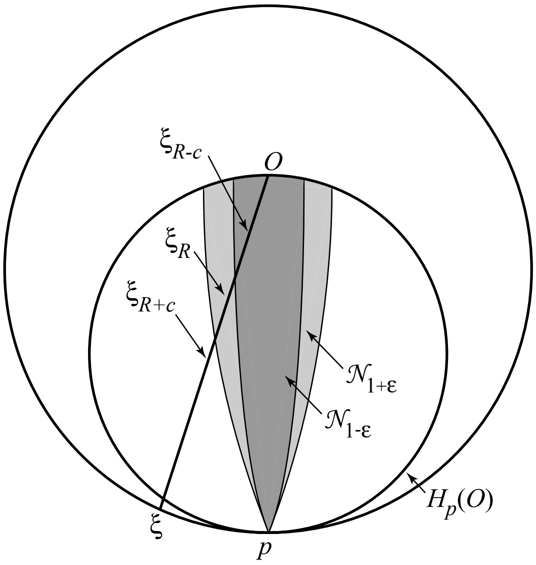

FIGURE 5. Here c is the constant from Lemma 4.6.

Note that

$$S(O,\xi,R+c)\subseteq S(O,\xi,R)\subseteq S(O,\xi,R-c)$$

Thus by Proposition 4.2 and Proposition 4.5, we have constants, C and C' such that

$$Ce^{-\delta(R+c)}(e^{\rho(\xi_{R+c},\Gamma O)})^{rank(\xi_R)-\delta}\leq \mu_O(S(O,\xi,R))\leq C'e^{-\delta(R-c)}(e^{\rho(\xi_{R-c},\Gamma O)})^{rank(\xi_R)-\delta}$$

But since c is independent of R and ξ and since

$$\rho(\xi_R,\Gamma O)-c\leq\rho(\xi_{R\pm c},\Gamma O)\leq\rho(\xi_R,\Gamma O)+c,$$

we have

$$e^{\rho(\xi_R,\Gamma O)}\asymp e^{\rho(\xi_{R\pm c},\Gamma O)}\quad\text{and}\quad e^{R\pm c}\asymp e^R,$$

and the result follows. ☐

5. COMPLETING THE PROOF OF THE GLOBAL MEASURE FORMULA

Let $O\in\mathcal{H}^n_{\mathbb{F}}$ be a fixed reference point and Γ a non-elementary geometrically finite group. Without loss of generality, let $O\in Hull(\Lambda_\Gamma)$. Let F be a convex fundamental domain for Γ containing O in its interior, and let P be the finite set of parabolic fixed points associated to F. In the previous section we proved our result for $p\in P$; here we will extend it to $q\in\Gamma P$. This part of the argument mimics that of Stratman and Velani [29] in the case $\mathbb{F}=\mathbb{R}$ without significant modification.

Let $\{\mathcal{H}_p : p\in P\}$ be a set of standard horoballs chosen to satisfy Equation (7) in Section 3.2, and let $\mathcal{H}_{\gamma p}=\gamma\mathcal{H}_p$. As we mentioned when we chose them, to prove the general global measure formula, we will need to shrink these. The following discussion leads up to that final choice of standard horoballs, which we will call $\mathcal{H}'_q$ for $q\in\Gamma P$.

The idea of the argument is to approximate the shadows of balls centered in $\mathcal{H}_q$ with the image under γ of those centered in $\mathcal{H}_p$, where γ is chosen so that

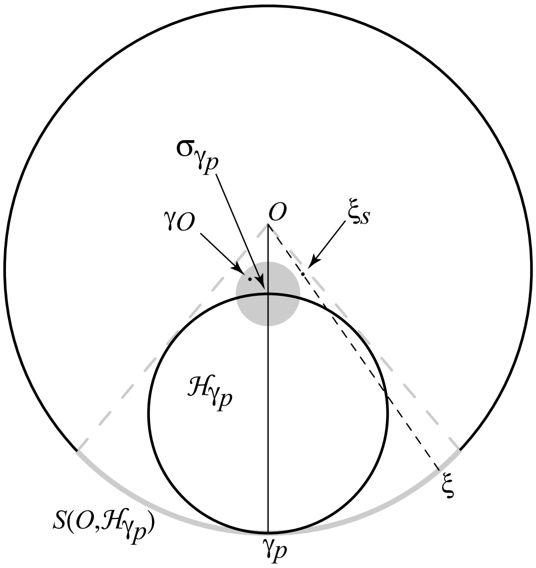

FIGURE 6. The grey disk represents the ball of radius c about $\sigma_{\gamma p}$ that by Lemma 5.1 contains γO. Here $s = \rho(O, \gamma O)$, as in Lemma 5.2.

$\gamma p = q$. Associate to each $\mathcal{H}_q$ with $q \in \Gamma P$ the point $\sigma_q \in \partial\mathcal{H}_q$ at the intersection of geodesic connecting O to q with the boundary of $\mathcal{H}_q$. Following Stratman and Velani, we call σ_q the "top" of $\mathcal{H}_q$. Since for each $q \in \Gamma P$, both q and O are in the convex hull of Λ_Γ, the point σ_q is as well. The following lemma says we can choose $\gamma \in \Gamma$ so that $\gamma p = q$ and γO lies a uniformly bounded distance from σ_q.

Lemma 5.1 ([29] Lemma 2.2)**.** *There is a positive constant c such that for each $p \in P$, and $\gamma \in \Gamma$, there is a $f \in \Gamma_{\gamma p}$ with*

$$\rho(\sigma_{\gamma p}, f\gamma O) \leq c.$$

Proof. Let $p \in P$, and let Γ_p be the stabilizer of p in Γ. Then $Hull(\Lambda_\Gamma) \cap \partial\mathcal{H}_p/\Gamma_p$ is compact. Let $F_p \subseteq \partial\mathcal{H}_p$ be a compact fundamental region for the action of Γ_p on $Hull(\Lambda_\Gamma) \cap \partial\mathcal{H}_p$ that contains σ_p, and let d_p be its diameter.

Let $\gamma \in \Gamma$. Then γF_p is a compact fundamental domain for the action of $\Gamma_{\gamma p}$ on $Hull(\Lambda_\Gamma) \cap \partial\mathcal{H}_{\gamma p}$ that contains $\gamma\sigma_p$. Since $\sigma_{\gamma p} \in Hull(\Lambda_\Gamma) \cap \partial\mathcal{H}_{\gamma p}$ as well, there is an element $f \in \Gamma_{\gamma p}$ such that $\sigma_{\gamma p} \in f\gamma F_p$. Then $\rho(\sigma_\gamma p, f\gamma\sigma_p) < d_p$, as both $f\gamma\sigma_p$ and $\sigma_{\gamma p}$ lie in $f\gamma F_p$. We have

$$\rho(\sigma_{\gamma p}, f\gamma O) \leq \rho(\sigma_{\gamma p}, f\gamma\sigma_p) + \rho(f\gamma\sigma_p, f\gamma O) < d_p + \rho(\sigma_p, O).$$

Let $c = max\{d_p + \rho(O, \sigma_p) : p \in P\}$, and the result follows. □

For each $p \in P$, let $\Gamma^p_{top} \subseteq \Gamma$ be a collection of $\gamma \in \Gamma$ such that for each $q \in \Gamma p$, $q = \gamma p$ for some $\gamma \in \Gamma^p_{top}$, and such that γO is near the top of $\mathcal{H}_q$, i.e. $\rho(\gamma O, \sigma_q) < c$, where c is the constant from Lemma 5.1. Let $S(O, \mathcal{H}_q)$ denote the endpoints in $\partial\mathcal{H}^n_{\mathbb{F}}$ of geodesics from O that pass though $\mathcal{H}_q$. This is convenient since $\xi \in S(O, \mathcal{H}_q)$ implies that there is an $R > 0$ such that $\xi_R \in \mathcal{H}_q$. We need to make the following uniform approximation.

Lemma 5.2. *There is a constant* $C > 0$ *such that for all* $\xi \in S(O, \mathcal{H}_q)$ *and all* $q \in \Gamma P$,

$$\rho(\xi_s, \gamma O) < C$$

where $\gamma \in \Gamma^p_{top}$ *satisfies* $q = \gamma p$, *and where* $s = \rho(O, \gamma O)$. *See Figure 6.*

Proof. Let $p \in P$ and let x be a point outside the standard horoball $\mathcal{H}_p$. For $\eta \in S(x, \mathcal{H}_p)$, let η^x_s be the point on the geodesic connecting x to η that lies a distance s from x. Let σ^x_p be the point of intersection of the geodesic connecting x to p and $\partial\mathcal{H}_p$. Then we prove we can find a constant $C > 0$ such that for all x in the complement of $\mathcal{H}_p$, and for all $\eta \in S(x, \mathcal{H}_p)$,

$$\rho(\sigma^x_p, \eta^x_s) < C,$$

where $s = \rho(x, \sigma^x_p)$.

We show this in the usual way for such uniform estimates: note that we can find a constant such that it holds for all x on any compact subset of $\mathcal{H}^n_{\mathbb{F}}$, and consider a sequence $\{x_i\}$ tending to $\partial\mathcal{H}^n_{\mathbb{F}}$. Let N be the nilpotent stabilizer of $\mathcal{H}_p$ in $O_{\mathbb{F}}(1, n)$. For each x_i, let $n_i \in N$ such that $n_i\sigma^{x_i}_p = \sigma^{x_1}_p$. Then the points n_ix_i tend to $\partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ along a single geodesic asymptotic to p. Let $x_\infty = \lim_{i\to\infty} n_ix_i \in \partial\mathcal{H}^n_{\mathbb{F}}\backslash\{p\}$ be the other endpoint of this geodesic. Let $s_i = \rho(x_i, \sigma^{x_i}_p)$, and let S_i be the pull back of the shadow $S(x_i, \mathcal{H}_p)$ along geodesics through x_i to the sphere of radius s_i about x_i. Note that for each $\eta \in S(x_i, \mathcal{H}_p)$, both $\eta^{x_i}_{s_i}$ and $\sigma^{x_i}_p$ lie in S_i, and hence $\rho(\sigma^{x_i}_p, \eta^{x_i}_{s_i})$ is bounded above by the diameter $diam(S_i)$. Now as $i \to \infty$, the sphere of radius s_i about n_ix_i tends to the horosphere based at x_∞ and through $\sigma^{x_1}_p$, and n_iS_i tends to the pull back to that horosphere along geodesics asymptotic to x_∞ of $S(x_\infty, \mathcal{H}_p)$, a set of finite diameter. Thus $diam(S_i) = diam(n_iS_i)$ is bounded for all i. Since P is finite, we can choose C large enough to satisfy this for all $p \in P$.

Now let $\xi \in S(O, \mathcal{H}_q)$, $q \in \Gamma P$ with $q = \gamma p$ for some $p \in P$ and $\gamma \in \Gamma^p_{top}$. Then $\xi = \gamma\eta$ for some $\eta \in S(\gamma^{-1}O, \mathcal{H}_p)$. Since $\gamma^{-1}O$ lies outside $\mathcal{H}_p$, the above estimate holds, and we have

$$\rho(\sigma_q, \xi_s) = \rho(\sigma^{\gamma^{-1}O}_p, \eta^{\gamma^{-1}O}_s) < C,$$

where $s = \rho(\gamma^{-1}O, \sigma^{\gamma^{-1}O}_p) = \rho(O, \sigma_q)$. Since by Lemma 5.1, γO is within a constant distance of σ_q and s is within a constant of $\rho(O, \gamma O)$, the result follows. □

Let $q = \gamma p \in \Gamma P$, with $\gamma \in \Gamma^p_{top}$ and $s = \rho(\gamma O, O)$. Let $\xi \in S(O, \mathcal{H}_q)$, then by Lemma 5.2 above, $\rho(\xi_s, \gamma O) < C$. Let $\eta = \gamma^{-1}\xi$. Then $\gamma\eta_t$ is the point on the geodesic connecting γO to ξ that lies a distance t from γO. Note that since asymptotic geodesics converge at least exponentially, we have

$$\rho(\gamma^{-1}\xi_{s+t}, \eta_t) < \rho(\xi_s, \gamma O)e^{-t} < Ce^{-t}, \tag{14}$$

for all $t > 0$.

The following two lemmas will allow us to uniformly make the desired approximation of shadows of balls centered in $\mathcal{H}_{\gamma p}$ with those centered in $\mathcal{H}_p$.

Lemma 5.3. *There exist* $u, v \in \mathbb{R}$ *such that for all* $\gamma \in \Gamma^p_{top}$ *and all* $\xi \in S(O, \mathcal{H}_{\gamma p})$

$$\gamma S(O, \gamma^{-1}\xi, t + u) \subseteq S(O, \xi, s + t) \subseteq \gamma S(O, \gamma^{-1}\xi, t - v)$$

where $s = \rho(O, \gamma O)$.

Proof. Note that $s = \rho(O, \gamma O)$ is bounded below since Γ is discrete. Let $0 < d \leq \rho(O, \gamma O)$ for all $\gamma \in \Gamma$. By Lemma 5.2, there is a constant C such that if $\gamma \in \Gamma^p_{top}$, and $\xi \in S(O, \mathcal{H}_{\gamma p})$, then $\rho(\xi_s, \gamma O) < C$. Applying γ^{-1}, we see that the geodesic connecting $\gamma^{-1}O$ to $\gamma^{-1}\xi$ enters $\mathcal{H}_p$ at a distance at most C from O. The set of geodesics that enter $\mathcal{H}_p$ within C of O is relatively compact; we describe how to choose u and v for a single such geodesic, and by compactness, we can make the choice uniformly for all of them.

Let $\xi, \zeta \in \partial\mathcal{H}^n_{\mathbb{F}}$ such that the geodesic from ζ to ξ enters $\mathcal{H}_p$ within C of O. Denote the parameterization of this geodesic by x_t, where $x_{+\infty} = \xi$, $x_{-\infty} = \zeta$, and let $x = x_0$ be any point on the geodesic lying a distance at least d from the point at which the geodesic enters $\mathcal{H}_p$. Let $s \geq d$ such that $x_s \in \partial\mathcal{H}_p$; then $\rho(x_s, O) < C$. Let ξ_t parameterize the geodesic from O to ξ, as usual. We want to find u and v such that for all $t > 1$,

$$S(O, \xi, t + u) \subseteq S(x, \xi, s + t) \subseteq S(O, \xi, t + v).$$

Note that we can do this for any compact subset of $t \in \mathbb{R}$; we show there exist u and v such that this holds for t large enough.

Let σ_t be the loxodromic isometry translating x_{t+s} to x_s and fixing ζ and ξ. Then as t goes to $+\infty$, $\sigma_t O$ tends to ζ, $\sigma_t \xi_t$ tends to x_s, and $\sigma_t(S(O, \xi, t))$ converges to the shadow from ζ of the ball $B(x_s, 1)$, which we will denote by $S(\zeta, B(x_s, 1))$. Let U be a fixed open neighborhood of ξ such that

$$\overline{U} \subseteq S(\zeta, B(x_s, 1)).$$

Then for t large enough, we have

$$\overline{U} \subseteq \sigma_t(S(O, \xi, t)).$$

Since as u increases, the sets $\sigma_t(S(O, \xi, t + u))$ form a descending nested sequence whose intersection is $\{\xi\}$, we can choose u such that for every t,

$$\sigma_t(S(O, \xi, t + u)) \subseteq U \subseteq S(\zeta, B(x_s, 1)).$$

Note that $\sigma_t S(x, \xi, s + t)$ are nested neighborhoods of ξ containing $S(\zeta, B(x_s, 1))$ in their intersection. Thus $\sigma_t(S(O, \xi, t + u)) \subseteq \sigma_t S(x, \xi, s + t)$, and $S(O, \xi, t + u) \subseteq S(x, \xi, s + t)$, as desired.

Now for each $v < s$, $\sigma_t S(O, \xi, t - v)$ converges to $S(\zeta, B(x_{s-v}, 1))$. Let $0 < v < d$. Then since as t increases, $\sigma_t S(x, \xi, t + s)$ is a descending nested sequence of sets whose intersection is $\{\xi\}$, there exists a t' such that,

$$\overline{\sigma_{t'} S(x, \xi, t' + s)} \subseteq S(\zeta, B(x_{s-v}, 1)).$$

Thus for t large enough, the convergence implies

$$\overline{\sigma_{t'} S(x, \xi, t' + s)} \subseteq \sigma_t S(O, \xi, t - v).$$

But for each $t > t'$, $\sigma_t S(x, \xi, t + s) \subseteq \sigma_{t'} S(x, \xi, t' + s)$ so we have, for large enough t, $S(x, \xi, t + s) \subseteq S(O, \xi, t - v)$, as desired. ☐

Lemma 5.4. *There exist positive constants C, C' such that for any $p \in P$, $\gamma \in \Gamma^p_{top}$, $\xi \in \Lambda_\Gamma$ and $R > 0$ such that $\xi_R \in \mathcal{H}_p$, we have*

$$C\mu_O(S(O, \xi, R))e^{-\delta\rho(O,\gamma O)} \leq \mu_O(\gamma(S(O, \xi, R))) \leq C'\mu_O(S(O, \xi, R))e^{-\delta\rho(O,\gamma O)}.$$

Proof. By the equivariance of μ_O, we have

$$\mu_O(\gamma(S(O,\xi,R))) = \mu_{\gamma^{-1}O}(S(O,\xi,R)) = \int_{S(O,\xi,R)} e^{-\delta b_\eta^O(\gamma^{-1}O)} d\mu_O[\eta], \tag{15}$$

so we study $b_\eta^O(\gamma^{-1}O)$.

Let $p \in P$. By Lemma 5.2, there is a constant $\hat{C}$ such that if $\gamma \in \Gamma_{top}^p$ and $\xi \in S(O, \mathcal{H}_p)$, then the geodesic connecting $\gamma^{-1}O$ to ξ enters $\mathcal{H}_p$ at a distance at most $\hat{C}$ from O. Let $A \subseteq \mathcal{H}_{\mathbb{F}}^n$ be the set of y such that the geodesic connecting y to a point of $S(O,\mathcal{H}_p)$ enters $\mathcal{H}_p$ at a distance at most $\hat{C}$ from O. Then $\gamma^{-1}O \in A$ for $\gamma \in \Gamma_{top}^p$. Fix $\epsilon > 0$. Then the elements $y \in A$ such that $b_\xi^O(y) \leq \epsilon$ for some $\xi \in S(O,\mathcal{H}_p)$ forms a relatively compact set. Thus since Γ is discrete, for all but finitely many $\gamma \in \Gamma_{top}^p$, we have $b_\xi^O(\gamma^{-1}O) > \epsilon$ for all $\xi \in \mathcal{H}_p$.

Let $x = x(\xi,\gamma)$ be the point of intersection of the geodesic connecting $\gamma^{-1}O$ to ξ and the horosphere $H_\xi(O)$. Then $\rho(O,x)$ is bounded for all $\xi \in S(O,\mathcal{H}_p)$ and $\gamma \in \Gamma_{top}^p$, and we can adjust $\hat{C}$ so that $\rho(x,O) \leq \hat{C}$ for all such x. Note that for all $\xi \in S(O,\mathcal{H}_p)$ and all but a finite number of $\gamma \in \Gamma_{top}^p$, $\rho(\gamma^{-1}O,x) = b_\xi^O(\gamma^{-1}O) > \epsilon$. By Lemma 2.1, there is a constant C depending only on ϵ such that

$$\rho(O,\gamma^{-1}O) + \rho(O,x) - C \leq \rho(\gamma^{-1}O,x) \leq \rho(O,\gamma^{-1}O) + \rho(O,x)$$

so

$$\rho(O,\gamma^{-1}O) - C < b_\xi^O(\gamma^{-1}O) < \rho(O,\gamma^{-1}O) + \hat{C},$$

for all $\xi \in S(O,\mathcal{H}_p)$ and we have constants c and c' such that

$$ce^{-\rho(O,\gamma O)} \leq e^{-b_\xi^O(\gamma^{-1}O)} \leq c'e^{-\rho(O,\gamma O)},$$

for all $\xi \in S(O,\mathcal{H}_p)$ and all $\gamma \in \Gamma_{top}^p$. The result now follows from Equation (15). □

Now we make our final choice of standard horospheres, beginning from those chosen in Section 3.2. Let C be the constant given in Lemma 5.2 and u and v be those given in Lemma 5.3. Let $\{\mathcal{H}'_q\}_{q\in\Gamma P}$ be the collection of horoballs, such that $\mathcal{H}'_q$ is the horoball based at q of that lies a distance $C + \max\{u,v\}$ from $\partial\mathcal{H}_q$ inside $\mathcal{H}_q$. We will prove the Global Measure Formula holds using these horoballs. Recall that our goal is to approximate the shadows of balls centered in $\mathcal{H}'_{\gamma p}$ with those centered in $\mathcal{H}_p$ for $p \in P$, where our results are known from our work in Section 4. This choice of horoballs $\mathcal{H}'_{\gamma p}$ ensures that when the ball whose shadow we are studying is centered in $\mathcal{H}'_{\gamma q}$, the balls we can use for our comparison given by Lemma 5.3 will indeed be centered in $\mathcal{H}_p$.

Finally, let $q \in \Gamma P$. Then there are $p \in P$ and $\gamma \in \Gamma_{top}^p$ such that $q = \gamma p$. Let $\xi \in \Lambda_\Gamma$ and $R > 0$ such that ξ_R lies in $\mathcal{H}'_q$. Let $u, v \in \mathbb{R}$ given by Lemma 5.3, such that

$$\gamma S(O,\gamma^{-1}\xi, R - s + u) \subseteq S(O,\xi,R) \subseteq \gamma S(O,\gamma^{-1}\xi, v + R - s)$$

where $s = \rho(O,\gamma O)$. Let $\eta = \gamma^{-1}\xi$.

First using Lemma 5.4, we have

$$\mu_O(\gamma S(O,\eta,R-s+u)) = e^{-\delta s}\mu_O(S(O,\eta,R-s+u)),$$

and

$$\mu_O(\gamma S(O,\eta,R-s-v)) = e^{-\delta s}\mu_O(S(O,\eta,R-s-v)).$$

Now let C be the constant given in Lemma 5.2. Note that by Equation (14), we have $\rho(\xi_R, \gamma\eta_{R-s}) < C$. Since $\xi_R \in \mathcal{H}'_q$ and since $\rho(\xi_R, \gamma\eta_{R-s-v}) < C + v$ and $\rho(\xi_R, \gamma\eta_{R-s+u}) < C + u$, we have $\gamma\eta_{R-s+u}, \gamma\eta_{R-s-v} \in \mathcal{H}_q$. Since $\rho(\eta_{R-s}, \Gamma O) = \rho(\gamma\eta_{R-s}, \Gamma O)$, by the triangle inequality, we have

$$\rho(\xi_R, \Gamma O) - C \leq \rho(\eta_{R-s}, \Gamma O) \leq \rho(\xi_R, \Gamma O) + C.$$

Thus we have $\eta_{R-s-v}, \eta_{R-s+u} \in \mathcal{H}_p$, and we may apply our results from Section 4 to calculate $\mu_O(S(O, \eta, R - s - v))$ and $\mu_O(S(O, \eta, R - s + u))$. Doing so directly yields

$$\mu_O(S(O, \xi, R)) \asymp e^{-\delta R}(e^{\rho(\xi_R, \Gamma O)})^{rank(\xi_R) - \delta},$$

as desired.

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