Dropping an Electron into a Ring of Charge

Christine Yap

Abstract

The behavior of dropping an electron into a ring of uniform charge is numerically approximated. By examining both linear and non-linear models under the influence of a damping force, it is shown that the electron continues to exhibit oscillatory behavior.

1 Introduction

In learning the fundamental theory of electromagnetism, the studies of the electric field of various configurations are examined. Calculating the electric field of a point charge above the center of a circular loop has been encountered by most students during their undergraduate studies of electrostatics, no doubt. This paper is aimed to further that study. Instead of having a fixed charged at some point above a circular loop, the charge is allowed to move freely and the behavior of this movement is approximated. *Mathematica* will be used for the numerical approximation. For the purposes of this paper, the electron is used as the point charge. Essentially, the electron will be "dropped" through the ring and its behavior anticipated. Figure 1 shows the setup of this theoretical experiment.

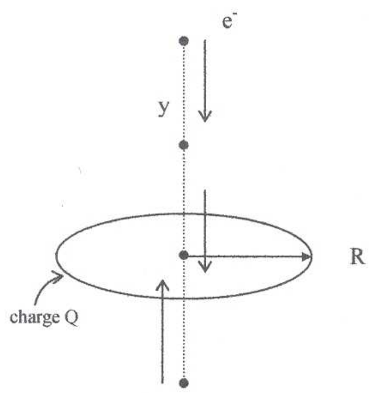

Figure 1 Model of Theoretical Experiment

2 Developing the Mathematical Model

2.1 The Force of the Electron Exerted

2 DEVELOPING THE MATHEMATICAL MODEL

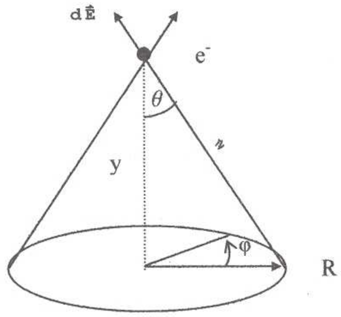

Figure 2

The separation vector is given by,

$$\mathscr{r}^2 = y^2(t) + R^2$$

The total charge on the ring is Q, and assuming the charge on the ring is uniform,

$$dq = \lambda\, dl$$

where λ is the charge per unit length of the ring and dl is an infinitesimal length along the ring. An element of the electric field is,

$$d\vec{E} = \frac{k\, dq}{\mathscr{r}^2}\,\hat{\mathscr{r}}$$

From the symmetry of the configuration, the only surviving component lies in the y direction which leads to,

$$d\vec{E} = \frac{k\,\lambda\, dl \cos\theta}{y^2(t) + R^2}\,\hat{y}$$

where,

$$\cos\theta = \frac{y}{\mathscr{r}}$$

so that,

$$dE = \frac{k\,\lambda\, y(t)\, dl}{(y^2(t) + R^2)^{3/2}}$$

Here, our infinitesimal line segment

$$dl = R\,d\phi$$

To find the electric field, we integrate along the circular loop, namely,

$$E = \int_0^{2\pi} \frac{k\,\lambda\, y(t)\, R}{(y^2(t) + R^2)^{3/2}}\, d\phi$$

$$= \frac{k\,\lambda\, y(t)\, R}{(y^2(t) + R^2)^{3/2}}\,(2\pi)$$

Recalling that,

$$\lambda = \frac{Q}{2\pi R}$$

$$\vec{E} = \frac{k\,Q\,y(t)}{(y^2(t) + R^2)^{3/2}}\,\hat{y}$$

This brings us to the expression for the force exerted from the electron on the ring of charge.

$$\vec{F} = -e^-\vec{E}$$

where e^- is simply the charge of the electron.

$$\vec{F} = -\frac{e^-\,k\,Q\,y(t)}{(y^2(t) + R^2)^{3/2}}\,\hat{y}$$

We finally have most of the tools we need for our purpose.

2.2 The Equation of Motion

Our goal is to find the equation of motion that our electron will satisfy. To achieve this we find the net force in the y-direction.

$$F_{NET,y} = F_{e^-}$$

$$m\ddot{y} = F_{e^-}$$

m being the electron mass.

$$\ddot{y} = \frac{F_{e^-}}{m}$$

$$\ddot{y} = -\frac{e^-\,k\,Q\,y(t)}{m\,(y^2(t) + R^2)^{3/2}} \qquad (1)$$

To make this equation linear, we use the approximation for small y(t), namely for y(t) << R.

Equation (1) becomes,

$$\ddot{y} = -\omega^2\, y(t) \tag{2}$$

where,

$$\omega^2 = \frac{e^-\, k\, Q}{m\, R^3} \tag{3}$$

With the system oscillating at frequency ω, the solution to the linear approximation is,

$$y(t) = A\cos(\omega t) \tag{4}$$

An illustration of the movement of the linear case is depicted in figure 3. This is simply the case of simple harmonic motion with initial condition of y(0) = 2.

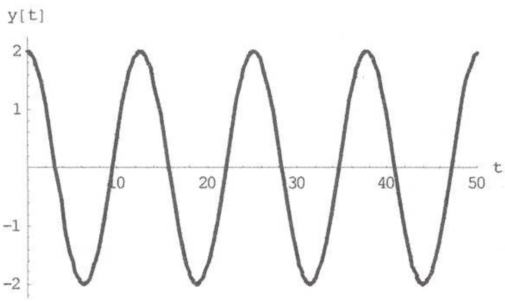

Figure 3 Simple Harmonic Motion

2.2.1 Radiation Damping

What happens in the case of damping? The damping we are referring to in this case is energy dissipation through radiative causes. For a more physical situation, energy will certainly not be

This in turn, leads to a modified equation of motion,

$$F_{NET,y} = F_{e^-} + F_r$$

$$\ddot{y}(t) = -\frac{e^-\, k\, Q}{m}\,\frac{y(t)}{(y^2(t) + R^2)^{3/2}} + \tau^*\,\dddot{y}(t)$$

From equation (3) and (4), it follows that,

$$\begin{aligned} y(t) &= A\cos(\omega t) \\ \dot{y}(t) &= -A\omega\sin(\omega t) \\ \ddot{y}(t) &= -A\omega^2\cos(\omega t) \\ \dddot{y}(t) &= A\omega^3\sin(\omega t) \\ &= -\omega^2\,\dot{y}(t) \end{aligned}$$

Our equation of motion with radiation damping is,

$$\ddot{y}(t) = -\omega^2\,\frac{y(t)}{\left(\frac{y^2(t)}{R^2} + 1\right)^{3/2}} - \omega^2\tau^*\,\dot{y}(t)$$

or equivalently,

$$\ddot{y}(t) + \omega^2\,\frac{y(t)}{\left(\frac{y^2(t)}{R^2} + 1\right)^{3/2}} - \omega^2\tau^*\,\dot{y}(t) = 0 \tag{5}$$

For the linear case,

$$\ddot{y}(t) = -\omega^2\,y(t) - \omega^2\tau^*\,\dot{y}(t) \tag{6}$$

The damping factor given by,

$$\gamma = \omega^2\tau^*$$

2.2.2 Dimensionless Equation of Motion

$$dy = R d\xi \qquad dt = \frac{1}{\omega} d\tau$$

$$d^2y = R d^2\xi \qquad (dt)^2 = \frac{1}{\omega^2} d\tau^2$$

The equation of motion (5) becomes,

$$\omega^2 R \frac{d^2\xi}{d\tau^2} + \frac{e^- kQ}{m} \frac{R\xi}{[(R\xi)^2 + R^2]^{3/2}} + \omega^3 \tau^* R \frac{d\xi}{d\tau} = 0$$

Simplifying the equation further,

$$\omega^2 R \frac{d^2\xi}{d\tau^2} + \frac{e^- kQ}{m R^3} \frac{R\xi}{(\xi^2 + 1)^{3/2}} + \omega^3 \tau^* R \frac{d\xi}{d\tau} = 0$$

$$\omega^2 R \frac{d^2\xi}{d\tau^2} + \omega^2 R \frac{\xi}{(\xi^2 + 1)^{3/2}} + \omega^3 \tau^* R \frac{d\xi}{d\tau} = 0$$

$$\frac{d^2\xi}{d\tau^2} + \frac{\xi}{(\xi^2 + 1)^{3/2}} + \omega\tau^* \frac{d\xi}{d\tau} = 0$$

2.3 The Numerical Approximation

Solving the above equation analytically is an arduous task, one which will not be attempted in the scope of this paper. However, a numerical solution using *Mathematica* allowing the approximation of the electron's behavior of the given system will be explored. This will be accomplished by examining various diagrams all constructed by means of a numerical approximation. It is important to mention that the numerical approximation utilized the following conditions:
$\xi(0) = 2$, $\xi'(0) = 0$, and $\omega\tau^* = 0.5$.

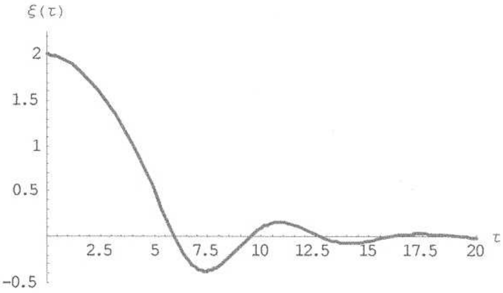

Figure 4 $\xi(\tau)$ versus τ

Figure 4 depicts the motion of the electron with time. It is useful to keep in mind the substitution made previously which allowed for our equation to become dimensionless. What is actually being observed in the diagram above is a plot of y/R versus ωt.

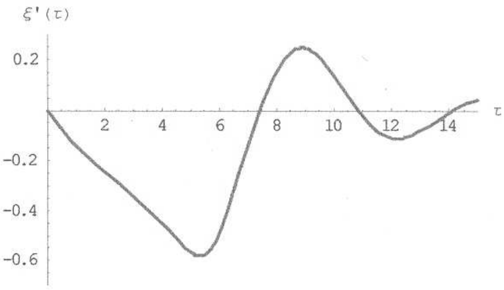

Figure 5 $\xi'(\tau)$ versus τ

Figure 5 depicts the velocity of the particle with time. The system started with zero velocity since the aim was to "drop" the electron into the ring. The negative increasing velocity shows the electron moving towards the ring.

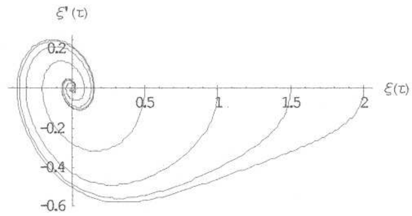

Figure 6 ξ'(τ) versus ξ(τ)

Figure 6 is quite interesting. It is a depiction of several trajectories of the electron given specific conditions, namely, $\xi(0) = 0, 0.5, 1, 1.5, 2$.

For comparative purposes, an illustration of the solution for the linear model with radiation damping is included. An attempt was made to be relatively consistent with the conditions set for the non-linear model. The conditions are: $y(0) = 2$, $y'(0) = 0$. Recalling equation (6) two constants must be accounted for, namely ω^2 and $\omega\tau^*$. The approximation utilizes $\omega^2 = 0.2$, and $\omega\tau^* = 0.5$.

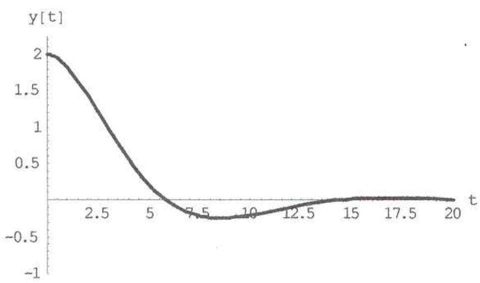

Figure 7 y(t) versus t

It is crucial to recall that the linear model was based on the approximation that $y(t) \ll R$. So when the same initial condition is used, that is, the starting point of the electron of two units, a system whose circular ring radius is much greater than two, is actually being observed.

Having noted the above, the behavior of the electron may now be explored. The motion of the electron in Figure 7 shows only slight oscillatory behavior. It is clear that the radiation damping plays a significant affect on the motion.

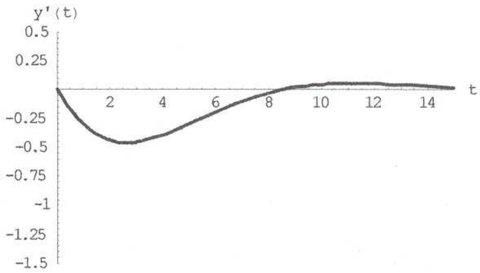

Figure 8 y'(t) versus t

The velocity of the electron is also affected by radiative damping. Figure 8 shows minimal oscillatory behavior also.

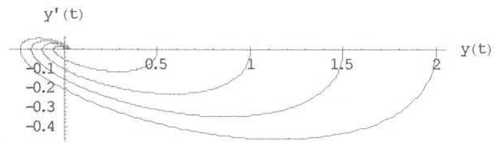

Figure 9 Trajectory of the Electron y'(t) versus y(t)

REFERENCES

3 Discussion

The next few images attempt to make a comparison the linear versus the non-linear approximation models. These images were produced using the numerical approximation method again with *Mathematica.* The same conditions were enforced on the system as before, the only addition was the radius of the ring where we utilized R = 1. Mention must be made that to accurately depict both images on the same scale, a dimensionless transformation of the non-linear model was not made in this case.

The function outlined below is named Y(T) which equivalently models the behavior of the electron. The purpose of a new name of the function was so as to not confuse the methods in which the functions were derived and to note that the graphs below are indeed on the same scale.

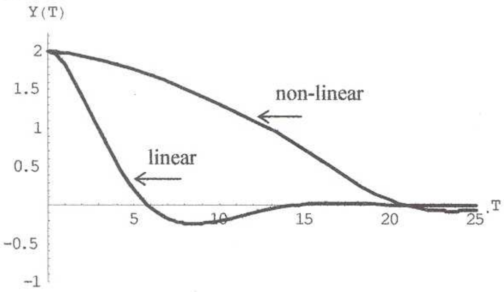

Figure 10 Comparative Motion Along the y-axis

There exists a noticeable difference in the motion of the particles. The linear model has a sharper slope.

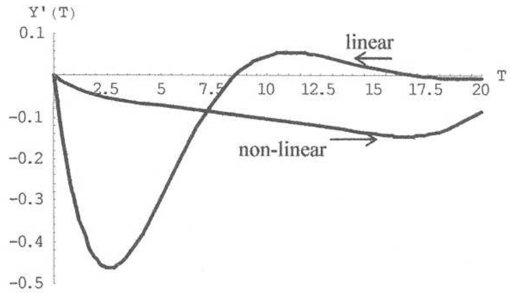

Figure 11 Comparative Velocities

The velocities exhibit very different characteristics.

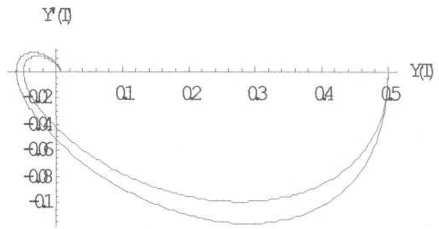

Figure 12 Comparative Trajectories

The trajectories depicted in Figure 12 are comparable. This is due to the fact that our initial conditions on both systems were equivalent. This behavior to say the least, is expected.

4 Conclusion

The numerical approximations of both the linear and non-linear models of dropping an electron into a ring of charge have showed quite interesting results. The most interesting outcome was clearly due to the radiation damping force that was included. This force was added by making an assumption that the system would not be perturbed to a great extent.

decrease in amplitude of the electron's motion. The amplitude of the electron's motion was decreased significantly, but to state steadily would not be an accurate depiction. The system appears to cycle shortly over 2 periods.

References

[1] H. Tahsiri: *Final Exam Project: Dropping an Electron into a Ring of Charge*, Physics 560A CSULB Handout Fall 2005.

[2] Griffiths, D. (1999). *Introduction to Electrodynamics*. New Jersey: Prentice Hall.