Newberger: Math 233, Spring 2004

Homework Aid on the Worksheet: Injections, Surjections and Bijections.

(1) (10 points) Page 137 #4.

For this problem, there are clever ways to explain why there are certain numbers of injections and surjections between these two sets, but you can always use bruit force. For example you could list all of the functions from S to T (there are 8) and all the functions from T to S (there are 9), and count the ones that are injective and surjective to answer the question.

(2) (10 points) Page 138 #9. In this problem you are asked for conditions that make something happen. Write your answers as if-then statements: "If [my condition holds], then [something happens]." (Up to 3 bonus points for a correct proof of your statements.)

If you followed these instructions, and stated a correct if-then statement, and I took off points for not providing proofs, come see me. I think I forgot what I asked you to do exactly when I graded this problem.

(3) (10 points) Determine whether or not each of the following functions are one-to-one, onto and/or a bijection. Prove your answers are correct.

(a) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $f(a, c) = (c, a)$.

This function is both one-to-one and onto. First let's make sure we understand what this function does. What is $f(1, 2)$*? You should get* $(2, 1)$.

One-to-one. As in all one-to-one proofs, we will assume x_1 *and* x_2 *are elements of* $\mathbb{R} \times \mathbb{R}$*, and prove the if-then statement: if* $f(x_1) = f(x_2)$*, then* $x_1 = x_2$*. In this problem, since* x_1 *and* x_2 *are elements of* $\mathbb{R} \times \mathbb{R}$*. This means they each have two coordinates. So since* $x_1, x_2 \in \mathbb{R} \times \mathbb{R}$*, we have* $x_1 = (s_1, t_1)$ *and* $x_2 = (s_2, t_2)$ *for some* $s_1, s_2, t_1, t_2 \in \mathbb{R}$*. So start by assuming* $f(x_1) = f(x_2)$*. Then substitute in* $x_1 = (s_1, t_1)$ *and* $x_2 = (s_2, t_2)$*, so you have* $f(s_1, t_1) = f(s_2, t_2)$*. Now calculate the lefthand and righthand side using the formula for* f*, and explain why we must have* $s_1 = s_2$ *and* $t_1 = t_2$*, so that* $x_1 = x_2$*,as desired.*

Onto. As in all onto proofs, we will begin with an element of the codomain. Let $y \in \mathbb{R} \times \mathbb{R}$*. Now our codomain has two coordinates, so this means we have* $y = (s, t)$ *for some* $s, t \in \mathbb{R}$*. Now find* x *in the domain of this function such that* $f(x) = y$*. Since the domain is* $\mathbb{R} \times \mathbb{R}$*, this means we want to find an* x *with two coordinates: i.e. we want to find* $(u, v) \in \mathbb{R} \times \mathbb{R}$ *such that* $f(u, v) = (s, t)$*. Then we let* $x = (u, v)$ *and we get* $f(x) = y$*, as desired.*

(b) $g : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $g(t) = (t, t)$.

This function is one-to-one, but not onto. First let's make sure that we understand this function. What is $g(3)$*? You should get* $(3, 3)$ *which is an element of the codomain* $\mathbb{R} \times \mathbb{R}$.

One-to-one. As in all one-to-one proofs, assume x_1 *and* x_2 *are elements of* $\mathbb{R}$*, and prove the if-then statement: if* $g(x_1) = g(x_2)$*, then* $x_1 = x_2$.

Onto. To show that g is not onto, find an element y of the codomain such that for all $x \in \mathbb{R}$, we have $g(x) \neq y$. Since our codomain is $\mathbb{R} \times \mathbb{R}$, y will have two coordinates. So you should find (u, v) such that for all $x \in \mathbb{R}$, we have $g(x) \neq (u, v)$.

(c) $j : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ given by $j(m, x) = (-1)^m x$.

This function is onto, but not one to one. First let's make sure that we understand the function. What is $j(4, -1)$ and what is $j(3, -3)$? You should get -1 and 3 respectively.

One-to-one. To show this function is not one-to-one, find a pair of elements x_1 and x_2 in the domain such that $j(x_1) = j(x_2)$, but $x_1 \neq x_2$. Since the domain is $\mathbb{Z} \times \mathbb{R}$, this means x_1 and x_2 will each have two coordinates, the first an integer and the second a real number. So you should find a particular numerical example of pairs (n_1, s_1) and (n_2, s_2) in $\mathbb{Z} \times \mathbb{R}$ such that $j(n_1, s_1) = j(n_2, s_2)$, but $(n_1, s_1) \neq (n_2, s_2)$.

(4) (10 points) Let $h : \mathbb{C} - \{0\} \to S^1$ be given by $h(z) = \frac{z}{|z|}$. Remember complex numbers are of the form $a + bi$ where $a, b \in \mathbb{R}$. We draw them in $\mathbb{R}^2$ by plotting $a + bi$ at (a, b).

(a) Draw two diagrams one showing the domain and the other the codomain of h (labelling your answers).

(b) Why do you think I chose the domain to be $\mathbb{C} - \{0\}$ rather than simply $\mathbb{C}$?

(c) Calculate $h(i)$, $h(\cos(t) + i\sin(t))$ where t is any real number, $h(2+3i)$, $h(-2i)$ and $h(5)$.

Remember how to calculate the modulus of a complex number: $|a + bi| = \sqrt{a^2 + b^2}$. So for example, $|i| = \sqrt{0^2 + 1^2} = 1$.

(d) Draw a graph of $\mathbb{R}^2$ showing i, 5, $2 + 3i$ and $-2i$. Label the image under h of each of these points on the graph. Draw and arrow connecting each of these points and its image.

(e) Is h onto? Prove your answer is correct.

This function is onto. As in all onto proofs, we will begin with an element of the codomain. Let $y \in S^1$. What does this mean? Remember

$$S^1 = \{z \in \mathbb{C} |\ |z| = 1\}.$$

So $y \in S^1$ means that $y \in \mathbb{C}$ and $|y| = 1$. Since $y \in \mathbb{C}$, $y = a + bi$ for some $a, b \in \mathbb{R}$. So we get $1 = |y| = \sqrt{a^2 + b^2}$. Now find an element x in the domain such that $h(x) = y$. Since the domain is $\mathbb{C} - \{0\}$, you should find a number $x \in \mathbb{C}$ such that $h(x) = y$. Try taking x as a multiple of y. For example take $x = 5y = 5a + 5bi$. Use the fact that $|y| = \sqrt{a^2 + b^2} = 1$ to show that $h(x) = y$. This works because if we think of the elements of $\mathbb{C}$ as vectors, the function h simply takes a vector z and normalizes it (making it length 1, so that it lands on the unit circle).

(f) Is h one-to-one? Prove your answer is correct.

This function is not one-to-one. To show this function is not one-to-one, find a pair of elements x_1 *and* x_2 *in the domain such that* $h(x_1) = h(x_2)$*, but* $x_1 \neq x_2$*. Since the domain is* $\mathbb{C} - \{0\}$*, both* x_1 *and* x_2 *are complex numbers. Look at the discussion above, under one-to-one. What do you get if you take* $x = 2y$ *instead of* $x = 5y$*? This should help you find* x_1 *and* x_2*.*